

# Contents

<i>Preface</i>	v
1. Finite-Horizon Models	1
1.1 Preliminaries . . . . .	1
1.2 Model Description . . . . .	3
1.3 Dynamic Programming Approach . . . . .	5
1.4 Examples . . . . .	8
1.4.1 Non-transitivity of the correlation . . . . .	8
1.4.2 The more frequently used control is not better . .	9
1.4.3 Voting . . . . .	11
1.4.4 The secretary problem . . . . .	13
1.4.5 Constrained optimization . . . . .	14
1.4.6 Equivalent Markov selectors in non-atomic MDPs	17
1.4.7 Strongly equivalent Markov selectors in non-atomic MDPs . . . . .	20
1.4.8 Stock exchange . . . . .	25
1.4.9 Markov or non-Markov strategy? Randomized or not? When is the Bellman principle violated? . .	27
1.4.10 Uniformly optimal, but not optimal strategy . . .	31
1.4.11 Martingales and the Bellman principle . . . . .	32
1.4.12 Conventions on expectation and infinities . . . . .	34
1.4.13 Nowhere-differentiable function $v_t(x)$ ; discontinuous function $v_t(x)$ . . . . .	38
1.4.14 The non-measurable Bellman function . . . . .	43
1.4.15 No one strategy is uniformly $\varepsilon$ -optimal . . . . .	44
1.4.16 Semi-continuous model . . . . .	46

2.	Homogeneous Infinite-Horizon Models: Expected Total Loss	51
2.1	Homogeneous Non-discounted Model . . . . .	51
2.2	Examples . . . . .	54
2.2.1	Mixed Strategies . . . . .	54
2.2.2	Multiple solutions to the optimality equation . . .	56
2.2.3	Finite model: multiple solutions to the optimality equation; conserving but not equalizing strategy .	58
2.2.4	The single conserving strategy is not equalizing and not optimal . . . . .	58
2.2.5	When strategy iteration is not successful . . . . .	61
2.2.6	When value iteration is not successful . . . . .	63
2.2.7	When value iteration is not successful: positive model I . . . . .	67
2.2.8	When value iteration is not successful: positive model II . . . . .	69
2.2.9	Value iteration and stability in optimal stopping problems . . . . .	71
2.2.10	A non-equalizing strategy is uniformly optimal . .	73
2.2.11	A stationary uniformly $\varepsilon$ -optimal selector does not exist (positive model) . . . . .	75
2.2.12	A stationary uniformly $\varepsilon$ -optimal selector does not exist (negative model) . . . . .	77
2.2.13	Finite-action negative model where a stationary uniformly $\varepsilon$ -optimal selector does not exist . . . .	80
2.2.14	Nearly uniformly optimal selectors in negative models . . . . .	83
2.2.15	Semi-continuous models and the blackmailer's dilemma . . . . .	85
2.2.16	Not a semi-continuous model . . . . .	88
2.2.17	The Bellman function is non-measurable and no one strategy is uniformly $\varepsilon$ -optimal . . . . .	91
2.2.18	A randomized strategy is better than any selector (finite action space) . . . . .	92
2.2.19	The fluid approximation does not work . . . . .	94
2.2.20	The fluid approximation: refined model . . . . .	97
2.2.21	Occupation measures: phantom solutions . . . . .	101
2.2.22	Occupation measures in transient models . . . . .	104
2.2.23	Occupation measures and duality . . . . .	107

2.2.24	Occupation measures: compactness . . . . .	109
2.2.25	The bold strategy in gambling is not optimal (house limit) . . . . .	112
2.2.26	The bold strategy in gambling is not optimal (inflation) . . . . .	115
2.2.27	Search strategy for a moving target . . . . .	119
2.2.28	The three-way duel (“Truel”) . . . . .	122
3.	Homogeneous Infinite-Horizon Models: Discounted Loss	127
3.1	Preliminaries . . . . .	127
3.2	Examples . . . . .	128
3.2.1	Phantom solutions of the optimality equation . .	128
3.2.2	When value iteration is not successful: positive model . . . . .	130
3.2.3	A non-optimal strategy $\hat{\pi}$ for which $v_x^{\hat{\pi}}$ solves the optimality equation . . . . .	132
3.2.4	The single conserving strategy is not equalizing and not optimal . . . . .	134
3.2.5	Value iteration and convergence of strategies . . .	135
3.2.6	Value iteration in countable models . . . . .	137
3.2.7	The Bellman function is non-measurable and no one strategy is uniformly $\varepsilon$ -optimal . . . . .	140
3.2.8	No one selector is uniformly $\varepsilon$ -optimal . . . . .	141
3.2.9	Myopic strategies . . . . .	141
3.2.10	Stable and unstable controllers for linear systems	143
3.2.11	Incorrect optimal actions in the model with partial information . . . . .	146
3.2.12	Occupation measures and stationary strategies . .	149
3.2.13	Constrained optimization and the Bellman principle . . . . .	152
3.2.14	Constrained optimization and Lagrange multipliers . . . . .	153
3.2.15	Constrained optimization: multiple solutions . . .	157
3.2.16	Weighted discounted loss and $(N, \infty)$ -stationary selectors . . . . .	158
3.2.17	Non-constant discounting . . . . .	160
3.2.18	The nearly optimal strategy is not Blackwell optimal . . . . .	163
3.2.19	Blackwell optimal strategies and opportunity loss	164

3.2.20	Blackwell optimal and $n$ -discount optimal strategies . . . . .	165
3.2.21	No Blackwell (Maitra) optimal strategies . . . . .	168
3.2.22	Optimal strategies as $\beta \rightarrow 1-$ and MDPs with the average loss – I . . . . .	171
3.2.23	Optimal strategies as $\beta \rightarrow 1-$ and MDPs with the average loss – II . . . . .	172
4.	Homogeneous Infinite-Horizon Models: Average Loss and Other Criteria . . . . .	177
4.1	Preliminaries . . . . .	177
4.2	Examples . . . . .	179
4.2.1	Why lim sup? . . . . .	179
4.2.2	AC-optimal non-canonical strategies . . . . .	181
4.2.3	Canonical triplets and canonical equations . . . . .	183
4.2.4	Multiple solutions to the canonical equations in finite models . . . . .	186
4.2.5	No AC-optimal strategies . . . . .	187
4.2.6	Canonical equations have no solutions: the finite action space . . . . .	188
4.2.7	No AC- $\varepsilon$ -optimal stationary strategies in a finite state model . . . . .	191
4.2.8	No AC-optimal strategies in a finite-state semi-continuous model . . . . .	192
4.2.9	Semi-continuous models and the sufficiency of stationary selectors . . . . .	194
4.2.10	No AC-optimal stationary strategies in a unichain model with a finite action space . . . . .	195
4.2.11	No AC- $\varepsilon$ -optimal stationary strategies in a finite action model . . . . .	198
4.2.12	No AC- $\varepsilon$ -optimal Markov strategies . . . . .	199
4.2.13	Singular perturbation of an MDP . . . . .	201
4.2.14	Blackwell optimal strategies and AC-optimality . . . . .	203
4.2.15	Strategy iteration in a unichain model . . . . .	204
4.2.16	Unichain strategy iteration in a finite communicating model . . . . .	207
4.2.17	Strategy iteration in semi-continuous models . . . . .	208
4.2.18	When value iteration is not successful . . . . .	211
4.2.19	The finite-horizon approximation does not work . . . . .	213

4.2.20	The linear programming approach to finite models	215
4.2.21	Linear programming for infinite models . . . . .	219
4.2.22	Linear programs and expected frequencies in finite models . . . . .	223
4.2.23	Constrained optimization . . . . .	225
4.2.24	AC-optimal, bias optimal, overtaking optimal and opportunity-cost optimal strategies: periodic model . . . . .	229
4.2.25	AC-optimal and average-overtaking optimal strategies . . . . .	232
4.2.26	Blackwell optimal, bias optimal, average-overtaking optimal and AC-optimal strategies . .	235
4.2.27	Nearly optimal and average-overtaking optimal strategies . . . . .	238
4.2.28	Strong-overtaking/average optimal, overtaking optimal, AC-optimal strategies and minimal opportunity loss . . . . .	239
4.2.29	Strong-overtaking optimal and strong*-overtaking optimal strategies . . . . .	242
4.2.30	Parrondo's paradox . . . . .	247
4.2.31	An optimal service strategy in a queueing system	249
Afterword		253
Appendix A Borel Spaces and Other Theoretical Issues		257
A.1	Main Concepts . . . . .	257
A.2	Probability Measures on Borel Spaces . . . . .	260
A.3	Semi-continuous Functions and Measurable Selection . . .	263
A.4	Abelian (Tauberian) Theorem . . . . .	265
Appendix B Proofs of Auxiliary Statements		267
Notation		281
List of the Main Statements		283
<i>Bibliography</i>		285
<i>Index</i>		291