

Contents

1	Introduction and Main Results	1
1.1	Formulation of the Problem.....	4
1.2	Statement of Main Results	13
1.3	Summary of the Contents	18
1.4	Notes and Comments.....	30
 Part I Elements of Analysis		
2	Elements of Probability Theory	35
2.1	Measurable Spaces and Functions	36
2.1.1	Measurable Spaces	36
2.1.2	Measures	40
2.1.3	Measurable Functions	43
2.2	Probability Spaces	47
2.3	Random Variables and Expectations	48
2.4	Independence	62
2.4.1	Independent Events	62
2.4.2	Independent Random Variables	63
2.4.3	Independent Algebras	65
2.5	Conditional Probabilities.....	71
2.6	Conditional Expectations	85
2.7	Notes and Comments.....	98
3	Elements of Functional Analysis	99
3.1	Linear Operators and Functionals	100
3.2	Quasinormed Linear Spaces	100
3.2.1	Bounded Sets	103
3.2.2	Continuity of Linear Operators	103
3.2.3	Topologies of Linear Operators	103
3.2.4	The Banach–Steinhaus Theorem.....	104
3.2.5	Product Spaces	105

3.3	Normed Linear Spaces	105
3.3.1	Finite Dimensional Spaces	109
3.3.2	The Hahn–Banach Extension Theorem.....	110
3.3.3	Dual Spaces	114
3.3.4	Annihilators	115
3.3.5	Dual Spaces of Normed Factor Spaces	116
3.3.6	Bidual Spaces	116
3.3.7	Weak Convergence	117
3.3.8	Weak* Convergence	117
3.3.9	Transposes	118
3.4	Continuous Functions and Measures	119
3.4.1	Spaces of Continuous Functions	119
3.4.2	Space of Signed Measures	122
3.4.3	The Riesz–Markov Representation Theorem	123
3.4.4	Weak Convergence of Measures	134
3.5	Closed Operators	135
3.6	Complemented Subspaces	136
3.7	Compact Operators	137
3.7.1	Definition and Basic Properties of Compact Operators	137
3.7.2	The Riesz–Schauder Theory	138
3.8	Fredholm Operators	140
3.8.1	Definition and Basic Properties of Fredholm Operators	140
3.8.2	Stability Theorem for Indices of Fredholm Operators ...	141
3.9	Notes and Comments.....	142
4	Theory of Semigroups	143
4.1	Banach Space Valued Functions	143
4.2	Operator Valued Functions.....	145
4.3	Exponential Functions	147
4.4	Contraction Semigroups	149
4.4.1	The Hille–Yosida Theory of Contraction Semigroups ...	149
4.4.2	The Contraction Semigroup Associated with the Heat Kernel.....	162
4.5	The Hille–Yosida Theory of (C_0) Semigroups.....	170
4.5.1	Semigroups and Their Infinitesimal Generators.....	170
4.5.2	Infinitesimal Generators and Their Resolvents.....	178
4.5.3	The Hille–Yosida Theorem.....	185
4.5.4	(C_0) Semigroups and Initial-Value Problems	192
4.6	Notes and Comments.....	196

Part II Elements of Partial Differential Equations

5	Theory of Distributions	201
5.1	Notation	202
5.1.1	Points in Euclidean Spaces	202
5.1.2	Multi-indices and Derivations	202
5.2	Function Spaces	203
5.2.1	L^p Spaces	203
5.2.2	Convolutions	204
5.2.3	Spaces of C^k Functions	205
5.2.4	The Space of Test Functions	207
5.2.5	Hölder Spaces	208
5.2.6	Friedrichs' Mollifiers	209
5.3	Differential Operators	212
5.4	Distributions and the Fourier Transform	212
5.4.1	Definitions and Basic Properties of Distributions	212
5.4.2	Topologies on $\mathcal{D}'(\Omega)$	218
5.4.3	The Support of a Distribution	218
5.4.4	The Dual Space of $C^\infty(\Omega)$	220
5.4.5	Tensor Products of Distributions	221
5.4.6	Convolutions of Distributions	223
5.4.7	The Jump Formula	225
5.4.8	Regular Distributions with Respect to One Variable	226
5.4.9	The Fourier Transform	228
5.4.10	Tempered Distributions	232
5.4.11	The Fourier Transform of Tempered Distributions	241
5.5	Operators and Kernels	254
5.5.1	Schwartz's Kernel Theorem	256
5.5.2	Regularizers	264
5.6	Layer Potentials	267
5.6.1	Single and Double Layer Potentials	267
5.6.2	The Green Representation Formula	268
5.7	Distribution Theory on a Manifold	271
5.7.1	Manifolds	271
5.7.2	Distributions on a Manifold	275
5.7.3	Differential Operators on a Manifold	277
5.7.4	Operators and Kernels on a Manifold	278
5.8	Domains of Class C^r	279
5.9	Notes and Comments	282
6	Sobolev and Besov Spaces	285
6.1	Hardy's Inequality	286
6.2	Sobolev Spaces	288
6.2.1	First Definition of Sobolev Spaces	288
6.2.2	Second Definition of Sobolev Spaces	289
6.2.3	Definition of General Sobolev Spaces	293

6.3	Definition of Besov Spaces on the Boundary	293
6.4	Trace Theorems	298
6.5	Notes and Comments.....	311
7	Theory of Pseudo-differential Operators	313
7.1	Manifolds with Boundary and the Double of a Manifold	314
7.2	Function Spaces	317
7.3	Fourier Integral Operators	320
7.3.1	Symbol Classes	320
7.3.2	Phase Functions.....	322
7.3.3	Oscillatory Integrals	324
7.3.4	Definitions and Basic Properties of Fourier Integral Operators	326
7.4	Pseudo-differential Operators.....	328
7.4.1	Definitions of Pseudo-differential Operators	328
7.4.2	Basic Properties of Pseudo-differential Operators	333
7.4.3	Pseudo-differential Operators on a Manifold.....	337
7.4.4	Hypoelliptic Pseudo-differential Operators	340
7.5	Potentials and Pseudo-differential Operators	341
7.6	The Transmission Property	344
7.7	The Boutet de Monvel Calculus	349
7.7.1	Trace, Potential and Singular Green Operators on the Half-Space \mathbf{R}_+^n	349
7.7.2	The Boutet de Monvel Algebra	351
7.8	Distribution Kernel of a Pseudo-differential Operator.....	355
7.9	Notes and Comments.....	358
8	Waldenfels Operators and Maximum Principles	361
8.1	Borel Kernels and Maximum Principles	361
8.1.1	Linear Operators having Positive Borel Kernel	365
8.1.2	Positive Borel Kernels and Pseudo-Differential Operators.....	388
8.2	Maximum Principles for Waldenfels Operators	390
8.2.1	The Weak Maximum Principle	392
8.2.2	The Strong Maximum Principle	394
8.2.3	The Hopf Boundary Point Lemma.....	401
8.3	Notes and Comments.....	408
Part III Markov Processes, Semigroups and Boundary Value Problems		
9	Markov Processes, Transition Functions and Feller Semigroups	411
9.1	Markov Processes	412
9.1.1	Definitions of Markov Processes	413
9.1.2	Transition Functions	419
9.1.3	Kolmogorov's Equations	426

9.1.4	Feller and C_0 Transition Functions	428
9.1.5	Path Functions of Markov Processes	431
9.1.6	Stopping Times	432
9.1.7	Definition of Strong Markov Processes	437
9.1.8	The Strong Markov Property and Uniform Stochastic Continuity	438
9.2	Feller Semigroups and Transition Functions	439
9.2.1	Definition of Feller Semigroups	439
9.2.2	Characterization of Feller Semigroups in Terms of Transition Functions	440
9.3	The Hille–Yosida Theory of Feller Semigroups	447
9.3.1	Generation Theorems for Feller Semigroups.....	448
9.3.2	Generation Theorems for Feller Semigroups in Terms of Maximum Principles	454
9.4	Infinitesimal Generators of Feller Semigroups on a Bounded Domain (i)	455
9.5	Infinitesimal Generators of Feller Semigroups on a Bounded Domain (ii)	464
9.6	Notes and Comments.....	475
10	Semigroups and Boundary Value Problems for Waldenfels Operators	477
10.1	Formulation of the Problem.....	478
10.2	A Generation Theorem for Feller Semigroups on a Bounded Domain	487
10.3	The Dirichlet Problem for Waldenfels Operators	489
10.3.1	Existence and Uniqueness Theorem for the Dirichlet Problem	489
10.3.2	Proof of Theorem 10.4	489
10.4	Construction of Feller Semigroups and Boundary Value Problems	493
10.4.1	Green Operators G_α^0 and Harmonic Operators H_α	493
10.4.2	Boundary Value Problems and Reduction to the Boundary	498
10.4.3	A Generation Theorem for Feller Semigroups in Terms of Green Operators	509
10.5	Proof of Theorem 1.2	510
10.5.1	Proof of Theorem 10.21	510
10.5.2	End of Proof of Theorem 1.2.....	523
10.6	Unique Solvability for Second-Order Pseudo-differential Operators.....	525
10.6.1	Fundamental Results for Second-Order Pseudo-differential Operators	525
10.6.2	Proof of Theorem 10.23	526

10.7	The Symbol of the First-Order Pseudo-differential Operator Π_α	534
10.8	Notes and Comments.....	544
11	Proof of Theorem 1.3	547
11.1	The Space $C_0(\bar{D} \setminus M)$	549
11.2	End of Proof of Theorem 1.3	550
11.3	Notes and Comments.....	561
12	Markov Processes Revisited	563
12.1	Basic Definitions and Properties of Markov Processes	563
12.2	Path-Continuity of Markov Processes	566
12.3	Path-Continuity of Markov Processes Associated with Semigroups	573
12.4	Examples of Diffusion Processes on a Bounded Domain	576
12.4.1	The Neumann Case	577
12.4.2	The Robin Case	583
12.4.3	The Oblique Derivative Case.....	586
12.5	Notes and Comments.....	589
13	Concluding Remarks	591
13.1	Existence and Uniqueness Theorems for Boundary Value Problems	592
13.2	Generation Theorems for Analytic Semigroups on a Bounded Domain	596
13.3	The Integro-Differential Operator Case	599
13.4	Notes and Comments.....	603
A	Boundedness of Pseudo-differential Operators	605
A.1	The Littlewood–Paley Series of a Tempered Distribution	605
A.2	Peetre’s Definition of Besov and Generalized Sobolev Spaces	607
A.3	Non-regular Symbols	610
A.4	The L^p Boundedness Theorem	615
A.5	Proof of Proposition A.11	617
A.6	Proof of Proposition A.12	623
A.6.1	Proof of the Case $\delta = 1$	624
A.6.2	Proof of the Case $0 \leq \delta < 1$	633
B	The Boutet de Monvel Calculus via Operator-Valued Pseudo-differential Operators	643
B.1	Introduction	643
B.2	Symbol Classes	645
B.2.1	General Notation.....	645
B.2.2	Group Actions	647
B.2.3	Operator-Valued Symbols	648
B.2.4	Duality	654
B.2.5	Wedge Sobolev Spaces	656

B.3	The Transmission Property	658
B.4	Symbol Classes for the Boutet de Monvel Calculus	670
B.4.1	The Operator ∂_+	670
B.4.2	Boundary Symbols on the Half-Space \mathbf{R}_+^n	670
B.5	The Analysis of Compositions	679
B.5.1	Decomposing $\text{Op}_{v_n}^+ p$	684
B.5.2	The Analysis of the Leftover Term	686
B.6	Operators on the Half-Space \mathbf{R}_+^n	692
B.6.1	Operators in the Boutet de Monvel Calculus	692
B.6.2	Outlook	699
References	701
Index	709