CONTENTS

Prefac	e	xv
Symbo	ols and conventions	xviii
0	Introduction	1
0.0	Preliminaries	1
0.0A	Types of equations considered	1
0.0B	Problems of uniqueness	2
0.0C	Fixed points	3
0.0D	General solution	4
0.0E	Solution depending on an arbitrary function	4
0.1	Special equations	5
0.1A	Change of variables	5
0.1 B	Schröder's, Abel's and Böttcher's equations	5
0.2	Applications	6
0.2A	Synthesizing judgements	6
0.2 B	Clock-graduation and the concept of chronon	9
0.2C	Sensation scale and Fechner's law	10
0.3	Iterative functional equations	11
1	Iteration	13
1.0	Introduction	13
1.1	Basic notions and some substantial facts	13
1.1A	Iterates, orbits and fixed points	13
1.1B	Limit points of the sequence of iterates	15
1.1C	Theorem of Šarkovskiĭ	16
1.1D	Attractive fixed points	17
1.2	Maximal domains of attraction	20

1.2A	Convergence of splinters	20
	Analytic mappings	22
1.3	The speed of convergence of iteration sequences	23
1.3A	Some lemmas	24
1.3B	Splinters behaving like geometric sequences	26
1.3C	Slower convergence of splinters	27
1.3D	Special cases	30
1.4	Iteration sequences of random-valued functions	32
1.4A	Preliminaries	33
1.4 B	Convergence of random splinters	35
1.5	Some fixed-point theorems	36
1.5A	Generalizations of the Banach contraction principle	37
1.5B	Case of product spaces	39
1.5C	Equivalence statement	42
1.6	Continuous dependence	43
1.7	Notes	46
2	Linear equations and branching processes	51
2.0	Introduction	51
2.1	Galton-Watson processes	52
2.1A	Probability generating functions	52
2.1B	Limit distributions	54
2.1C	Stationary measures for processes with immigration	56
2.1D	Restricted stationary measures for simple processes	58
2.2	Nonnegative solutions	59
2.2A	Negative g	60
2.2B	Positive g	60
2.3	Monotonic solutions	61
2.3A	Homogeneous equation	61
2.3B	Special inhomogeneous equation	64
2.3C	General inhomogeneous equation	65
2.3D	An example	66
2.3E	Homogeneous difference equation	68
2.3F	Schröder's equation	68
2.4	Convex solutions	69
2.4A	Lemmas	69
2.4B	Existence-and-uniqueness result	71
2.4C	A difference equation	72
2.4D	Abel's and Schröder's equations	73
2.5	Regularly varying solutions	74
2.5A	Regularly varying functions	75
2.5B	Homogeneous equation	75

.

vi

Con	tent	s
con		.0

2.5C	Special inhomogeneous equation	78
	Application to branching processes	81
	Conditional limit probabilities	81
	Stationary measures	83
	Restricted stationary measures	86
	Convex solutions of higher order	89
	Definitions and results	89
	A characterization of polynomials	91
	Notes	91
3	Regularity of solutions of linear equations	96
3.0	Introduction	96
3.1	Continuous solutions	96
3.1A	Homogeneous equation	97
3.1B	General continuous solution of the homogeneous equation	99
3.1C	Inhomogeneous equation	101
3.2	Continuous dependence of continuous solutions on given	
	functions	106
3.3	Asymptotic properties of solutions	108
3.3A	Solutions continuous at the origin	108
3.3B	Sample proofs	111
3.3C	Asymptotic series expansions	115
3.3D	Solutions discontinuous at the origin	117
3.4	Differentiable solutions	119
3.5	Special equations	122
3.5A	Schröder's equation	122
3.5B	Julia's equation	124
3.5C	Abel's equation	127
3.5D	A characterization of the cross ratio	129
3.6	Solutions of bounded variation	130
3.6A	Preliminaries	130
3.6B	Homogeneous equation	132
3.6C	Inhomogeneous equation	134
3.6D	Solutions of almost bounded variation	135
3.7	Applications	137
3.7A	An Anosov diffeomorphism without invariant measure	137
3.7 B	Doubly stochastic measures supported on a hairpin	138
3.7C	Phase and dispersion for second-order differential equations	141
3.8	Notes	143
	Analytic and integrable solutions of linear equations	148
4.0	Introduction	148

vii

4 1	Lincon equation in a tenalegical anges	140
	Linear equation in a topological space $\int f(x) dx = 1$	148
	Analytic solutions: the case $ f'(0) \leq 1$	150
	Extension theorems	150
	Existence and uniqueness results	152
	Continuous dependence on the data	154
	Analytic solutions: the case $ f'(0) = 1$	155
	The Siegel set	156
	Special inhomogeneous equation	156
	General homogeneous and inhomogeneous equations	159
	Analytic solutions: the case $f'(0) = 0$	160
4.5	Meromorphic solutions	165
4.6	Special equations	167
4.6A	The Schröder equation	167
4.6B	The Abel equation	168
4.7	Integrable solutions	169
4 .7 <i>A</i>	Preliminaries	170
4.7B	A functional inequality	172
4 .7C	Homogeneous equation	174
4.7D	Inhomogeneous equation	174
4 .7E	Lebesgue measure	175
4.8	Absolutely continuous solutions	177
4.8A	Existence-and-uniqueness result	177
4.8B	A Goursat problem	178
4.9	Notes	180
5	Theory of nonlinear equations	185
	Introduction	185
5.1	An extension theorem	185
5.2	Existence and uniqueness of continuous solutions	187
	Lipschitzian h	187
5.2B	Continuous dependence on the data	190
	Non-Lipschitzian h	190
	Existence via solutions of inequalities	192
	Continuous solution depending on an arbitrary function	194
	Extension of solutions	195
	Nonuniqueness theorems	195
	Existence theorems	197
	Comparison with the linear case	197
	Asymptotic properties of solutions	200
	Coincidence and existence theorems	200
	Solutions differentiable at the origin	200
	Lipschitzian solutions	203 204
5.5	Sipositizium solutions	204

viii

5.5A	Existence and uniqueness	205
5.5B	Lipschitzian Nemytskiĭ operators	206
	Smooth solutions	208
5.6A	Preliminaries	208
5.6 B	Existence and uniqueness of C^r solutions	210
5.6C	Lack of uniqueness of C ^r solutions	214
5.7	Local analytic solutions	216
5.7A	Unique solution	217
5.7B	Continuous dependence on the data	219
5.8	Equations in measure space	221
5.8A	Existence and uniqueness of L ^p solutions	222
5.8B	L^1 solutions	225
5.8C	Extension theorems	226
5.8D	L ^p solution depending on an arbitrary function	228
5.9	Notes	229
6	Equations of higher orders and systems of linear equations	235
	Introduction	235
	Particular solutions of some special equations	236
	Cauchy functional equations on a curve	236
	The Gaussian normal distribution	230
	Equation of Nth order	237
	Proofs	239
	Further applications	240
	Invariant measures under piecewise linear transformations	240
	Decomposition of two-place functions	243
	Cyclic equations	244
	Homogeneous equation with a finite group of substitutions	245
	Compatibility	246
	General solution	247
6.4	Matrix equation with constant g	248
	Reduction to systems of scalar equations	248
	Real solutions when some characteristic roots of g are	
	complex numbers	249
6.4C	Special system of two linear equations	250
	Discussion of more general cases	252
	Local C^{∞} solutions of a matrix equation	253
	Preliminaries	253
6.5B	Existence of a unique solution	254
	Solution depending on an arbitrary function	255
	Two existence theorems	257
6.6	Smooth solutions of the equation of Nth order	258

6.6A	Continuous and differentiable solutions	259
	Integrable solutions	260
6.7	Equation of Nth order with iterates of one function	260
6.7 <i>A</i>	Reduction of order	261
6.7 B	Constant coefficients	262
6.7C	Reduction to a matrix linear equation	262
6.8	Linear recurrence inequalities	263
6.8A	System of inequalities	263
6.8B	Consequences for single inequalities	264
6.9	Notes	265
7	Equations of infinite order and systems of nonlinear equations	270
7.0	Introduction	270
7.1	Extending solutions	271
7.1A	General extension theorem	271
7.1B	Two sufficient conditions	273
7.1C	Extending of continuous solutions	275
7.2	Existence and uniqueness	276
7.2A	Basic result	276
7.2B	Important special case	279
7.2C	An application	280
7.2D	Lipschitzian solutions	282
7.2E	Denumerable order	285
7.3	Stability	286
7.3A	Main result	286
7.3B	Special results	287
7.3C	Comments	290
7.4	Approximate solutions	291
7. 4 A	Two special equations	291
7. 4B	Approximation in Buck's sense	293
7.4C	Uniform approximation of a continuous mapping by a	
	Lipschitzian one	296
7.4D	Polynomial approximate solutions	298
7.5	Continuous dependence	299
7.5A	A general result	299
7.5 B	Important special case	301
7.6	A survey of results on systems of nonlinear equations of	
	finite orders	302
7.6A	Continuous solutions of h-systems	304
7.6 B	Solutions of h-systems with a prescribed asymptotic behaviour	305
7.6C	Differentiable solutions of h-systems	307
7.6D	Analytic solutions of h-systems	308

х

7.6E	Integrable solutions of h-systems	308
7.6F	Continuous solutions of g-systems	310
7.6G	Differentiable solutions of g-systems	311
7.7	Three sample proofs	313
7.8	Continuous solutions – deeper uniqueness conditions	319
7.8A	A crucial inequality	320
7.8 B	Result for the testing equation	322
7.8C	Proof of Theorem 7.8.1	324
7.8D	Uniqueness implies existence	325
7.9	Notes	327
8	On conjugacy	332
8.0	Introduction	332
	Conjugacy	333
8.1A	Change of variables	333
	Properties of the conjugacy relation	333
8.2	Linearization	334
8.2A	The Schröder equation	335
8.2B	Unique local C^r solutions in \mathbb{R}^N	336
8.2C	Further results on smooth solutions	339
8.3	The Böttcher equation	339
8.3A	Complex case	339
8.3B	Asymptotic behaviour and regularity of real solutions	341
8.4	Conjugate functions	342
8.4A	N-dimensional case	342
	One-dimensional case	343
8.5	Conjugate formal series and analytic functions	345
8.5A	Julia's equation and the iterative logarithm	346
	Formally conjugate power series	347
8.5C	Conjugate analytic functions	350
8.5D	Abel's equation	351
8.6	Permutable functions	353
8.7	Commuting formal series and analytic functions	355
	Formal power series that commute with a given one	355
8.7 <i>B</i>	Convergence of formal power series having iterative logarithm	357
8.7 <i>C</i>	Permutable analytic functions	358
8.7D	Conjugacy again	359
8.8	Notes	360
0	More on the Schröder and Abel equations	365
	More on the Schröder and Abel equations Introduction	365
		365
9.1	Principal solutions	202

Contents

9.2	The pre-Schröder system	368
	An equivalent system and automorphic functions	368
	Equivalence of the Schröder equation and the pre-Schröder	
	system	369
9.3	Abel-Schröder systems and the associativity equation	371
	Strict Archimedean associative functions	371
9.3B	Abel and Schröder equations jointly	373
9.3C	Existence of generators	374
9.3D	A solution of the Abel-Schröder system	375
9.4	Abel systems and differential equations with deviations	378
9.4A	A group of transformations	379
9.4B	Systems of simultaneous Abel equations	380
9.5	A Schröder system and a characterization of norms	382
9.5A	Characterization of norms	382
9.5B	A system of simultaneous Schröder equations	383
9.6	Notes	385
10	Characterization of functions	389
10.0	Introduction	389
10.1	Power functions	389
10.1A	Identity	389
10.1 B	Reciprocal	390
10.1C	Roots	391
10.1D	Comments	392
10.2	Logarithmic and exponential functions	393
10.2A	Logarithm	394
10.2 B	Exponential functions	395
10.2C	An improper integral	396
10.2D	Comments	398
10.3	Trigonometric and hyperbolic functions	399
10.3A	Cosine and hyperbolic cosine	399
10.3 B	Periodic solutions of the cosine equation	399
10.3C	Sine	403
10.3D	An improper integral	404
10.3E	Comments	406
10.4	Euler's gamma function	407
10.4A	The fundamental functional equation	407
10.4 B	Riemann-integrable solutions of an auxiliary equation	409
10.4C	Gauss's multiplication theorem	410
10.4D	Complex gamma function	412
10.4E	Legendre's duplication formula and Euler's functional equation	412
10.4F	Logarithmic derivative of the gamma function	414

	٠	٠	٠
v	1	1	1
л	1		1

10.5	Continuous nowhere differentiable functions	414
10.5A	The Weierstrass c.n.d. functions	414
10.5B	A characterization of S_p by homogeneous equations	415
	An inhomogeneous equation for S_p	415
	Comments	418
10.6	Notes	419
11	Iterative roots and invariant curves	421
11.0	Introduction	421
11.1	Purely set-theoretical case	422
11.2	Continuous and monotonic solutions	423
11.2A	Strictly increasing continuous iterative roots	424
11.2 B	Strictly decreasing roots of strictly decreasing functions	425
11.2C	Strictly decreasing roots of strictly increasing functions	427
11.3	Monotonic C^r solutions	428
11.4	C^1 iterative roots with nonzero multiplier	430
11.4A	A function with no smooth convex square roots	430
11.4 B	Necessary conditions	431
11.4C	Main existence theorem	432
11.4D	Convex and concave iterative roots	436
11.5	C^1 iterative roots with zero multiplier	438
11.5A	Abundance of solutions	438
	Convergence problem	440
11.5C	Uniqueness conditions	442
11.6	Complex domain and local analytic solutions	443
	Existence and uniqueness	444
11.6 B	Multiplier 1	445
11.6C	Multiplier being a primitive root of unity	447
	Fractional iterates of the roots of identity function	449
	Babbage equation and involutions	450
11.7A	Decreasing involutions	451
11.7 B	Primitive iterative roots, homographies	453
	Another characterization of reciprocals	455
	Dubikajtis' theorem	455
11.8 B	Volkmann's theorem	457
11.9	Invariant curves	459
11.9A	Simplifications and assumptions	460
	Equation of invariant curves and its Lipschitzian solutions	460
	Lack of uniqueness of Lipschitzian solutions	463
	Comments	465
11.9E	Euler's and other special equations	466
	Notes	458

12	Linear iterative functional inequalities	472
12.0	Introduction	472
12.1	$\{f\}$ -monotonic functions	473
12.2	Inequalities in the uniqueness case for associated equations	474
12.2A	Comparison theorems	475
12.2B	Representation theorems	476
12.3	Asymptotic behaviour of nonnegative CSs y of the	
	inhomogeneous inequality	477
12.3A	Some properties of y	477
12.3B	Unique solution of the associated equation	479
12.3C	Asymptotic behaviour of y	479
12.4	Regular solutions of the homogeneous inequality	481
12.4A	Estimates	481
12.4B	Regular solution	483
12.4C	Comparison theorems	484
12.4D	Representation theorems	485
12.5	The inhomogeneous inequality in the nonuniqueness case	488
12.5A	The best lower bound and the regular solution	488
12.5B	Properties of solutions of the inequality	489
12.5C	Representation theorem	490
12.6	Regular solutions of the homogeneous inequality determined	
	by asymptotic properties	490
12.6A	One-parameter family of solutions of the associated equation	491
12.6B	Regular solutions of the inequality	492
12.6C	Special behaviour of given functions	492
12.6D	Conditions equivalent to regularity	494
12.7	A homogeneous inequality of second order	495
12.7A	An equivalent system	495
12.7B	A property of the particular solution	497
12.7C	Reduction of order	498
12.8	An inequality of infinite order	499
12.9	Notes	501
Refere	nces	504
Autho	r index	546
Subjec	et index	549

xiv