## CONTENTS

Preface ..... xv
Symbols and conventions ..... xviii
0 Introduction ..... 1
0.0 Preliminaries ..... 1
0.0A Types of equations considered ..... 1
0.0B Problems of uniqueness ..... 2
0.0C Fixed points ..... 3
0.0D General solution ..... 4
0.0E Solution depending on an arbitrary function ..... 4
0.1 Special equations ..... 5
0.1A Change of variables ..... 5
0.1B Schröder's, Abel's and Böttcher's equations ..... 5
0.2 Applications ..... 6
0.2A Synthesizing judgements ..... 6
0.2B Clock-graduation and the concept of chronon ..... 9
0.2C Sensation scale and Fechner's law ..... 10
0.3 Iterative functional equations ..... 11
1 Iteration ..... 13
1.0 Introduction ..... 13
1.1 Basic notions and some substantial facts ..... 13
1.1 A Iterates, orbits and fixed points ..... 13
1.1B Limit points of the sequence of iterates ..... 15
1.1C Theorem of Šarkovskiĭ ..... 16
1.1D Attractive fixed points ..... 17
1.2 Maximal domains of attraction ..... 20
1.2A Convergence of splinters ..... 20
1.2B Analytic mappings ..... 22
1.3 The speed of convergence of iteration sequences ..... 23
1.3A Some lemmas ..... 24
1.3B Splinters behaving like geometric sequences ..... 26
1.3C Slower convergence of splinters ..... 27
1.3D Special cases ..... 30
1.4 Iteration sequences of random-valued functions ..... 32
1.4A Preliminaries ..... 33
1.4B Convergence of random splinters ..... 35
1.5 Some fixed-point theorems ..... 36
1.5A Generalizations of the Banach contraction principle ..... 37
1.5B Case of product spaces ..... 39
1.5C Equivalence statement ..... 42
1.6 Continuous dependence ..... 43
1.7 Notes ..... 46
2 Linear equations and branching processes ..... 51
2.0 Introduction ..... 51
2.1 Galton-Watson processes ..... 52
2.1 A Probability generating functions ..... 52
2.1B Limit distributions ..... 54
2.1C Stationary measures for processes with immigration ..... 56
2.1D Restricted stationary measures for simple processes ..... 58
2.2 Nonnegative solutions ..... 59
2.2A Negative g ..... 60
2.2B Positive g ..... 60
2.3 Monotonic solutions ..... 61
2.3 A Homogeneous equation ..... 61
2.3B Special inhomogeneous equation ..... 64
2.3C General inhomogeneous equation ..... 65
2.3D An example ..... 66
2.3E Homogeneous difference equation ..... 68
2.3F Schröder's equation ..... 68
2.4 Convex solutions ..... 69
2.4A Lemmas ..... 69
2.4B Existence-and-uniqueness result ..... 71
2.4C A difference equation ..... 72
2.4D Abel's and Schröder's equations ..... 73
2.5 Regularly varying solutions ..... 74
2.5A Regularly varying functions ..... 75
2.5B Homogeneous equation ..... 75
2.5C Special inhomogeneous equation ..... 78
2.6 Application to branching processes ..... 81
2.6A Conditional limit probabilities ..... 81
2.6B Stationary measures ..... 83
2.6C Restricted stationary measures ..... 86
2.7 Convex solutions of higher order ..... 89
2.7A Definitions and results ..... 89
2.7B A characterization of polynomials ..... 91
2.8 Notes ..... 91
3 Regularity of solutions of linear equations ..... 96
3.0 Introduction ..... 96
3.1 Continuous solutions ..... 96
3.1 A Homogeneous equation ..... 97
3.1B General continuous solution of the homogeneous equation ..... 99
3.1C Inhomogeneous equation ..... 101
3.2 Continuous dependence of continuous solutions on given functions ..... 106
3.3 Asymptotic properties of solutions ..... 108
3.3A Solutions continuous at the origin ..... 108
3.3B Sample proofs ..... 111
3.3C Asymptotic series expansions ..... 115
3.3D Solutions discontinuous at the origin ..... 117
3.4 Differentiable solutions ..... 119
3.5 Special equations ..... 122
3.5A Schröder's equation ..... 122
3.5B Julia's equation ..... 124
3.5C Abel's equation ..... 127
3.5D A characterization of the cross ratio ..... 129
3.6 Solutions of bounded variation ..... 130
3.6A Preliminaries ..... 130
3.6B Homogeneous equation ..... 132
3.6C Inhomogeneous equation ..... 134
3.6D Solutions of almost bounded variation ..... 135
3.7 Applications ..... 137
3.7 A An Anosov diffeomorphism without invariant measure ..... 137
3.7B Doubly stochastic measures supported on a hairpin ..... 138
3.7C Phase and dispersion for second-order differential equations ..... 141
3.8 Notes ..... 143
4 Analytic and integrable solutions of linear equations ..... 148
4.0 Introduction ..... 148
4.1 Linear equation in a topological space ..... 148
4.2 Analytic solutions: the case $\left|f^{\prime}(0)\right| \leqslant 1$ ..... 150
4.2A Extension theorems ..... 150
4.2B Existence and uniqueness results ..... 152
4.2C Continuous dependence on the data ..... 154
4.3 Analytic solutions: the case $\left|f^{\prime}(0)\right|=1$ ..... 155
4.3A The Siegel set ..... 156
4.3B Special inhomogeneous equation ..... 156
4.3C General homogeneous and inhomogeneous equations ..... 159
4.4 Analytic solutions: the case $f^{\prime}(0)=0$ ..... 160
4.5 Meromorphic solutions ..... 165
4.6 Special equations ..... 167
4.6A The Schröder equation ..... 167
4.6B The Abel equation ..... 168
4.7 Integrable solutions ..... 169
4.7A Preliminaries ..... 170
4.7B A functional inequality ..... 172
4.7C Homogeneous equation ..... 174
4.7D Inhomogeneous equation ..... 174
4.7E Lebesgue measure ..... 175
4.8 Absolutely continuous solutions ..... 177
4.8A Existence-and-uniqueness result ..... 177
4.8B A Goursat problem ..... 178
4.9 Notes ..... 180
5 Theory of nonlinear equations ..... 185
5.0 Introduction ..... 185
5.1 An extension theorem ..... 185
5.2 Existence and uniqueness of continuous solutions ..... 187
5.2A Lipschitzian h ..... 187
5.2B Continuous dependence on the data ..... 190
5.2C Non-Lipschitzian h ..... 190
5.2D Existence via solutions of inequalities ..... 192
5.3 Continuous solution depending on an arbitrary function ..... 194
5.3A Extension of solutions ..... 195
5.3B Nonuniqueness theorems ..... 195
5.3C Existence theorems ..... 197
5.3D Comparison with the linear case ..... 199
5.4 Asymptotic properties of solutions ..... 200
5.4A Coincidence and existence theorems ..... 200
5.4B Solutions differentiable at the origin ..... 203
5.5 Lipschitzian solutions ..... 204
5.5A Existence and uniqueness ..... 205
5.5B Lipschitzian Nemytskiĭ operators ..... 206
5.6 Smooth solutions ..... 208
5.6A Preliminaries ..... 208
5.6B Existence and uniqueness of $C^{r}$ solutions ..... 210
5.6C Lack of uniqueness of $C^{r}$ solutions ..... 214
5.7 Local analytic solutions ..... 216
5.7A Unique solution ..... 217
5.7B Continuous dependence on the data ..... 219
5.8 Equations in measure space ..... 221
5.8A Existence and uniqueness of $L^{p}$ solutions ..... 222
5.8B $L^{1}$ solutions ..... 225
5.8C Extension theorems ..... 226
$5.8 D L^{p}$ solution depending on an arbitrary function ..... 228
5.9 Notes ..... 229
6 Equations of higher orders and systems of linear equations ..... 235
6.0 Introduction ..... 235
6.1 Particular solutions of some special equations ..... 236
6.1 A Cauchy functional equations on a curve ..... 236
6.1B The Gaussian normal distribution ..... 237
6.1C Equation of Nth order ..... 237
6.1D Proofs ..... 239
6.2 Further applications ..... 240
6.2A Invariant measures under piecewise linear transformations ..... 240
6.2B Decomposition of two-place functions ..... 243
6.3 Cyclic equations ..... 244
6.3A Homogeneous equation with a finite group of substitutions ..... 245
6.3B Compatibility ..... 246
6.3C General solution ..... 247
6.4 Matrix equation with constant $g$ ..... 248
6.4A Reduction to systems of scalar equations ..... 248
6.4B Real solutions when some characteristic roots of $g$ are complex numbers ..... 249
6.4C Special system of two linear equations ..... 250
6.4D Discussion of more general cases ..... 252
6.5 Local $C^{\infty}$ solutions of a matrix equation ..... 253
6.5A Preliminaries ..... 253
6.5B Existence of a unique solution ..... 254
6.5C Solution depending on an arbitrary function ..... 255
6.5D Two existence theorems ..... 257
6.6 Smooth solutions of the equation of $N$ th order ..... 258
6.6A Continuous and differentiable solutions ..... 259
6.6B Integrable solutions ..... 260
6.7 Equation of $N$ th order with iterates of one function ..... 260
6.7A Reduction of order ..... 261
6.7B Constant coefficients ..... 262
6.7C Reduction to a matrix linear equation ..... 262
6.8 Linear recurrence inequalities ..... 263
6.8A System of inequalities ..... 263
6.8B Consequences for single inequalities ..... 264
6.9 Notes ..... 265
7 Equations of infinite order and systems of nonlinear equations ..... 270
7.0 Introduction ..... 270
7.1 Extending solutions ..... 271
7.1A General extension theorem ..... 271
7.1B Two sufficient conditions ..... 273
7.1C Extending of continuous solutions ..... 275
7.2 Existence and uniqueness ..... 276
7.2A Basic result ..... 276
7.2B Important special case ..... 279
7.2C An application ..... 280
7.2D Lipschitzian solutions ..... 282
7.2E Denumerable order ..... 285
7.3 Stability ..... 286
7.3A Main result ..... 286
7.3B Special results ..... 287
7.3C Comments ..... 290
7.4 Approximate solutions ..... 291
7.4A Two special equations ..... 291
7.4B Approximation in Buck's sense ..... 293
7.4C Uniform approximation of a continuous mapping by a Lipschitzian one ..... 296
7.4D Polynomial approximate solutions ..... 298
7.5 Continuous dependence ..... 299
7.5A A general result ..... 299
7.5B Important special case ..... 301
7.6 A survey of results on systems of nonlinear equations of finite orders ..... 302
7.6A Continuous solutions of h-systems ..... 304
7.6B Solutions of h-systems with a prescribed asymptotic behaviour ..... 305
7.6C Differentiable solutions of h-systems ..... 307
7.6D Analytic solutions of $h$-systems ..... 308
7.6E Integrable solutions of $h$-systems ..... 308
7.6F Continuous solutions of $g$-systems ..... 310
7.6G Differentiable solutions of $g$-systems ..... 311
7.7 Three sample proofs ..... 313
7.8 Continuous solutions - deeper uniqueness conditions ..... 319
7.8A A crucial inequality ..... 320
7.8B Result for the testing equation ..... 322
7.8C Proof of Theorem 7.8.1 ..... 324
7.8D Uniqueness implies existence ..... 325
7.9 Notes ..... 327
8 On conjugacy ..... 332
8.0 Introduction ..... 332
8.1 Conjugacy ..... 333
8.1A Change of variables ..... 333
8.1 B Properties of the conjugacy relation ..... 333
8.2 Linearization ..... 334
8.2A The Schröder equation ..... 335
8.2B Unique local $C^{r}$ solutions in $\mathbb{R}^{N}$ ..... 336
8.2C Further results on smooth solutions ..... 339
8.3 The Böttcher equation ..... 339
8.3A Complex case ..... 339
8.3B Asymptotic behaviour and regularity of real solutions ..... 341
8.4 Conjugate functions ..... 342
8.4A N -dimensional case ..... 342
8.4B One-dimensional case ..... 343
8.5 Conjugate formal series and analytic functions ..... 345
8.5A Julia's equation and the iterative logarithm ..... 346
8.5B Formally conjugate power series ..... 347
8.5C Conjugate analytic functions ..... 350
8.5D Abel's equation ..... 351
8.6 Permutable functions ..... 353
8.7 Commuting formal series and analytic functions ..... 355
8.7A Formal power series that commute with a given one ..... 355
8.7B Convergence of formal power series having iterative logarithm ..... 357
8.7C Permutable analytic functions ..... 358
8.7D Conjugacy again ..... 359
8.8 Notes ..... 360
9 More on the Schröder and Abel equations ..... 365
9.0 Introduction ..... 365
9.1 Principal solutions ..... 365
9.2 The pre-Schröder system ..... 368
9.2A An equivalent system and automorphic functions ..... 368
9.2B Equivalence of the Schröder equation and the pre-Schröder system ..... 369
9.3 Abel-Schröder systems and the associativity equation ..... 371
9.3A Strict Archimedean associative functions ..... 371
9.3B Abel and Schröder equations jointly ..... 373
9.3C Existence of generators ..... 374
9.3D A solution of the Abel-Schröder system ..... 375
9.4 Abel systems and differential equations with deviations ..... 378
9.4A A group of transformations ..... 379
9.4B Systems of simultaneous Abel equations ..... 380
9.5 A Schröder system and a characterization of norms ..... 382
9.5 A Characterization of norms ..... 382
9.5B A system of simultaneous Schröder equations ..... 383
9.6 Notes ..... 385
10 Characterization of functions ..... 389
10.0 Introduction ..... 389
10.1 Power functions ..... 389
10.1 A Identity ..... 389
10.1B Reciprocal ..... 390
10.1C Roots ..... 391
10.1D Comments ..... 392
10.2 Logarithmic and exponential functions ..... 393
10.2A Logarithm ..... 394
10.2B Exponential functions ..... 395
10.2C An improper integral ..... 396
10.2D Comments ..... 398
10.3 Trigonometric and hyperbolic functions ..... 399
10.3A Cosine and hyperbolic cosine ..... 399
10.3B Periodic solutions of the cosine equation ..... 399
10.3C Sine ..... 403
10.3D An improper integral ..... 404
10.3E Comments ..... 406
10.4 Euler's gamma function ..... 407
10.4A The fundamental functional equation ..... 407
10.4B Riemann-integrable solutions of an auxiliary equation ..... 409
10.4C Gauss's multiplication theorem ..... 410
10.4D Complex gamma function ..... 412
10.4E Legendre's duplication formula and Euler's functional equation ..... 412
10.4F Logarithmic derivative of the gamma function ..... 414
10.5 Continuous nowhere differentiable functions ..... 414
10.5 A The Weierstrass c.n.d. functions ..... 414
10.5B A characterization of $S_{p}$ by homogeneous equations ..... 415
10.5C An inhomogeneous equation for $S_{p}$ ..... 415
10.5D Comments ..... 418
10.6 Notes ..... 419
11 Iterative roots and invariant curves ..... 421
11.0 Introduction ..... 421
11.1 Purely set-theoretical case ..... 422
11.2 Continuous and monotonic solutions ..... 423
11.2A Strictly increasing continuous iterative roots ..... 424
11.2B Strictly decreasing roots of strictly decreasing functions ..... 425
11.2C Strictly decreasing roots of strictly increasing functions ..... 427
11.3 Monotonic $C^{r}$ solutions ..... 428
11.4 $C^{1}$ iterative roots with nonzero multiplier ..... 430
11.4A A function with no smooth convex square roots ..... 430
11.4B Necessary conditions ..... 431
11.4C Main existence theorem ..... 432
11.4D Convex and concave iterative roots ..... 436
$11.5 C^{1}$ iterative roots with zero multiplier ..... 438
11.5A Abundance of solutions ..... 438
11.5B Convergence problem ..... 440
11.5C Uniqueness conditions ..... 442
11.6 Complex domain and local analytic solutions ..... 443
11.6A Existence and uniqueness ..... 444
11.6B Multiplier 1 ..... 445
11.6C Multiplier being a primitive root of unity ..... 447
11.6D Fractional iterates of the roots of identity function ..... 449
11.7 Babbage equation and involutions ..... 450
11.7A Decreasing involutions ..... 451
11.7B Primitive iterative roots, homographies ..... 453
11.8 Another characterization of reciprocals ..... 455
11.8A Dubikajtis' theorem ..... 455
11.8B Volkmann's theorem ..... 457
11.9 Invariant curves ..... 459
11.9A Simplifications and assumptions ..... 460
11.9B Equation of invariant curves and its Lipschitzian solutions ..... 460
11.9C Lack of uniqueness of Lipschitzian solutions ..... 463
11.9D Comments ..... 465
11.9E Euler's and other special equations ..... 466
11.10 Notes ..... 458
12 Linear iterative functional inequalities ..... 472
12.0 Introduction ..... 472
$12.1\{f\}$-monotonic functions ..... 473
12.2 Inequalities in the uniqueness case for associated equations ..... 474
12.2A Comparison theorems ..... 475
12.2B Representation theorems ..... 476
12.3 Asymptotic behaviour of nonnegative CSs $y$ of the inhomogeneous inequality ..... 477
12.3A Some properties of $y$ ..... 477
12.3B Unique solution of the associated equation ..... 479
12.3C Asymptotic behaviour of $y$ ..... 479
12.4 Regular solutions of the homogeneous inequality ..... 481
12.4A Estimates ..... 481
12.4B Regular solution ..... 483
12.4C Comparison theorems ..... 484
12.4D Representation theorems ..... 485
12.5 The inhomogeneous inequality in the nonuniqueness case ..... 488
12.5A The best lower bound and the regular solution ..... 488
12.5B Properties of solutions of the inequality ..... 489
12.5C Representation theorem ..... 490
12.6 Regular solutions of the homogeneous inequality determined by asymptotic properties ..... 490
12.6A One-parameter family of solutions of the associated equation ..... 491
12.6B Regular solutions of the inequality ..... 492
12.6C Special behaviour of given functions ..... 492
12.6D Conditions equivalent to regularity ..... 494
12.7 A homogeneous inequality of second order ..... 495
12.7 A An equivalent system ..... 495
12.7B A property of the particular solution ..... 497
12.7C Reduction of order ..... 498
12.8 An inequality of infinite order ..... 499
12.9 Notes ..... 501
References ..... 504
Author index ..... 546
Subject index ..... 549

