

Contents

1. ALGEBRAIC FOUNDATIONS

| | |
|--|----|
| 1.1. Sets | 1 |
| 1.1.1. The notion of set | 1 |
| 1.1.2. The specification of sets | 1 |
| 1.1.3. Subsets | 2 |
| 1.1.4. Union and intersection | 3 |
| 1.1.5. Difference and complement | 5 |
| 1.1.6. Ordered pairs, and products of sets | 5 |
| 1.1.7. Sets of sets | 5 |
| Exercises | 7 |
| 1.2. Functions | 8 |
| 1.2.1. The notion of function | 8 |
| 1.2.2. Injective, surjective, and bijective functions | 9 |
| 1.2.3. Inverse functions | 9 |
| 1.2.4. Composition of functions | 10 |
| 1.3. Binary and n -ary operations | 10 |
| 1.3.1. Binary operations on a set | 10 |
| 1.3.2. Idempotent, commutative, and associative operations | 11 |
| 1.3.3. Unit and null elements | 13 |
| 1.3.4. Cancellation | 15 |
| 1.3.5. Distributivity | 16 |
| 1.3.6. n -ary operations | 16 |
| Exercises | 17 |
| 1.4. Binary relations | 17 |
| 1.4.1. The concept of relation | 17 |
| 1.4.2. Complementary and converse relations | 18 |
| 1.4.3. Some special kinds of relations on a set | 19 |
| 1.4.4. Equivalence relations | 19 |
| 1.4.5. Orderings | 21 |
| Exercises | 23 |

| | |
|---|----|
| 1.5. Lattices | 23 |
| 1.5.1. Introduction | 23 |
| 1.5.2. Algebraic definition of a semilattice and lattice | 24 |
| 1.5.3. The principle of duality for lattices | 27 |
| 1.5.4. Some further properties of lattices | 27 |
| 1.5.5. Distributive lattices and complemented lattices | 28 |
| 1.5.6. Boolean algebras | 30 |
| Exercises | 31 |
| | |
| 2. GRAPHS AND ALGORITHMS | |
| 2.1. Introduction | 32 |
| 2.2. Graphs | 32 |
| 2.2.1. Definition of a graph | 32 |
| 2.2.2. Graphs and relations | 35 |
| 2.2.3. Simple graphs | 37 |
| 2.2.4. p -graphs and multigraphs | 39 |
| 2.3. Paths on graphs | 40 |
| 2.3.1. Paths and cycles | 40 |
| 2.3.2. Chains on simple graphs | 42 |
| 2.4. Some forms of connectedness of graphs | 43 |
| 2.4.1. Accessibility | 43 |
| 2.4.2. Connectivity | 43 |
| 2.4.3. Strong connectivity | 44 |
| 2.4.4. Algorithms for finding accessible sets and the components of a graph | 45 |
| 2.5. Acyclic graphs | 48 |
| 2.6. Trees | 53 |
| 2.6.1. Elementary properties of trees | 53 |
| 2.6.2. Transverse orderings of trees | 57 |
| 2.6.3. An algorithm for traversing trees | 62 |
| 2.7. Backtrack programming (or 'tree-search') algorithms | 64 |
| 2.7.1. The determination of elementary paths | 65 |
| 2.7.2. A general description of backtrack programming | 70 |

| | |
|--|-----|
| 2.7.3. The determination of Hamiltonian cycles | 71 |
| 2.8. The time complexity of algorithms | 75 |
| Exercises | 81 |
| Additional notes and bibliography | 82 |
| | |
| 3. PATH PROBLEMS | |
| | |
| 3.1. Introduction | 84 |
| 3.2. An algebra for path problems | 84 |
| 3.2.1. Definition of a path algebra | 84 |
| 3.2.2. Some examples of the path algebra | 85 |
| 3.2.3. Elementary properties of path algebras | 88 |
| 3.2.4. The solution of equations | 92 |
| 3.2.5. Matrices | 93 |
| 3.3. Labelled graphs | 95 |
| 3.3.1. Definition of a labelled graph | 95 |
| 3.3.2. Path labels | 95 |
| 3.3.3. Absorptive graphs | 96 |
| 3.4. Graphs and matrices | 97 |
| 3.4.1. Adjacency matrices | 97 |
| 3.4.2. Powers of matrices | 99 |
| 3.4.3. Stable matrices | 102 |
| 3.4.4. Absorptive matrices | 103 |
| 3.5. The formulation and solution of path problems | 105 |
| 3.6. Direct methods of solution | 108 |
| 3.6.1. Triangular matrices | 108 |
| 3.6.2. The Gauss elimination method | 113 |
| 3.6.3. The Jordan elimination method | 119 |
| 3.7. Iterative methods | 123 |
| 3.7.1. The Jacobi, Gauss–Seidel, and double-sweep iterative methods | 123 |
| 3.7.2. Conditions for validity of the iterative methods | 127 |
| 3.7.3. Comparison of the iterative methods | 130 |
| 3.8. A special method for totally ordered path algebras | 132 |

| | |
|---|-----|
| 3.9. Practical considerations | 134 |
| 3.9.1. Implementation of the path-finding algorithms | 134 |
| 3.9.2. Comparison of the algorithms | 135 |
| Exercises | 136 |
| Additional notes and bibliography | 140 |
| | |
| 4. CONNECTIVITY | |
| 4.1. Introduction | 142 |
| 4.2. Separation by the removal of arcs | 142 |
| 4.2.1. Separating arc sets | 142 |
| 4.2.2. Cut sets of arcs | 142 |
| 4.2.3. The determination of proper separating arc sets | 144 |
| 4.2.4. Basic arcs | 146 |
| 4.3. Basis graphs | 148 |
| 4.4. Separation by the removal of nodes | 150 |
| 4.4.1. Separating node sets | 150 |
| 4.4.2. Separating nodes | 151 |
| 4.5. Edge separation on simple graphs | 153 |
| 4.5.1. Separating edge sets | 153 |
| 4.5.2. Cut sets of edges | 154 |
| 4.5.3. The determination of proper separating edge sets | 155 |
| 4.5.4. Bridges | 156 |
| 4.6. Spanning trees in simple graphs | 158 |
| 4.6.1. Free trees | 158 |
| 4.6.2. Spanning trees | 160 |
| 4.6.3. Shortest spanning trees | 160 |
| 4.6.4. Determination of the spanning trees of a graph | 163 |
| 4.7. Node separation on simple graphs | 164 |
| 4.7.1. Articulation sets | 165 |
| 4.7.2. Articulation nodes | 165 |
| Exercises | 171 |
| Additional notes and bibliography | 173 |

| | |
|--|-----|
| 5. INDEPENDENT SETS, DOMINATING SETS, AND COLORATIONS | |
| 5.1. Introduction | 175 |
| 5.2. Independent sets | 175 |
| 5.2.1. Independent node sets | 175 |
| 5.2.2. Cliques | 177 |
| 5.2.3. Independent edge sets (matchings) | 181 |
| 5.3. Dominating sets | 187 |
| 5.4. Colorations | 188 |
| 5.4.1. Node colorations | 188 |
| 5.4.2. An algorithm for colouring a graph | 192 |
| 5.4.3. Edge colorations | 195 |
| Exercises | 196 |
| Additional notes and bibliography | 196 |
| 6. FLOWS IN NETWORKS | |
| 6.1. Introduction | 198 |
| 6.2. Networks | 198 |
| 6.3. Network flows | 199 |
| 6.3.1. Definition of a network flow | 199 |
| 6.3.2. Operations on flows | 200 |
| 6.3.3. Elementary flows | 200 |
| 6.3.4. Arc capacities | 201 |
| 6.4. Displacement networks | 202 |
| 6.4.1. The notion of a displacement network | 202 |
| 6.4.2. Flows on displacement networks | 203 |
| 6.4.3. Flow differences | 204 |
| 6.5. Maximal flows in networks | 206 |
| 6.5.1. The maximal flow problem | 206 |
| 6.5.2. Cuts | 210 |
| 6.5.3. An algorithm for constructing maximal flows | 212 |
| 6.6. Minimal-cost maximal flows | 214 |
| 6.6.1. Minimal-cost flows | 214 |
| 6.6.2. Costs on displacement networks | 214 |
| 6.6.3. Cost-reducing cycles | 216 |

| | |
|--|------------|
| 6.6.4. Algorithms for constructing minimal-cost maximal flows | 217 |
| 6.7. Transportation and assignment problems | 221 |
| 6.7.1. The transportation problem | 221 |
| 6.7.2. The optimal assignment problem | 223 |
| 6.7.3. The assignment of machines to a fixed schedule of tasks | 224 |
| 6.8. Circulations | 226 |
| 6.8.1. Definition of a circulation | 226 |
| 6.8.2. Auxiliary networks | 226 |
| 6.8.3. The construction of circulations | 229 |
| 6.8.4. Minimal-cost circulations | 231 |
| 6.9. Practical considerations | 232 |
| 6.9.1. Implementation of the flow-augmentation method | 233 |
| 6.9.2. Implementation of the cost-reduction method | 234 |
| 6.9.3. Comparison of methods | 234 |
| Exercises | 235 |
| Additional notes and bibliography | 240 |
| SOLUTIONS TO SELECTED EXERCISES | 242 |
| BIBLIOGRAPHY | 256 |
| SUBJECT INDEX | 269 |
| AUTHOR INDEX | 275 |