

---

# Contents

Preface	ix
Notation	xiii
Frequently Used Functions	xvii
Chapter 1. Preliminary Notions	1
§1.1. Approximating a sum by an integral	1
§1.2. The Euler-MacLaurin formula	2
§1.3. The Abel summation formula	5
§1.4. Stieltjes integrals	7
§1.5. Slowly oscillating functions	8
§1.6. Combinatorial results	9
§1.7. The Chinese Remainder Theorem	10
§1.8. The density of a set of integers	11
§1.9. The Stirling formula	11
§1.10. Basic inequalities	13
Problems on Chapter 1	15
Chapter 2. Prime Numbers and Their Properties	19
§2.1. Prime numbers and their polynomial representations	19
§2.2. There exist infinitely many primes	21
§2.3. A first glimpse at the size of $\pi(x)$	21
§2.4. Fermat numbers	22
§2.5. A better lower bound for $\pi(x)$	24

§2.6. The Chebyshev estimates	24
§2.7. The Bertrand Postulate	29
§2.8. The distance between consecutive primes	31
§2.9. Mersenne primes	32
§2.10. Conjectures on the distribution of prime numbers	33
Problems on Chapter 2	36
Chapter 3. The Riemann Zeta Function	39
§3.1. The definition of the Riemann Zeta Function	39
§3.2. Extending the Zeta Function to the half-plane $\sigma > 0$	40
§3.3. The derivative of the Riemann Zeta Function	41
§3.4. The zeros of the Zeta Function	43
§3.5. Euler's estimate $\zeta(2) = \pi^2/6$	45
Problems on Chapter 3	48
Chapter 4. Setting the Stage for the Proof of the Prime Number Theorem	51
§4.1. Key functions related to the Prime Number Theorem	51
§4.2. A closer analysis of the functions $\theta(x)$ and $\psi(x)$	52
§4.3. Useful estimates	53
§4.4. The Mertens estimate	55
§4.5. The Möbius function	56
§4.6. The divisor function	58
Problems on Chapter 4	60
Chapter 5. The Proof of the Prime Number Theorem	63
§5.1. A theorem of D. J. Newman	63
§5.2. An application of Newman's theorem	65
§5.3. The proof of the Prime Number Theorem	66
§5.4. A review of the proof of the Prime Number Theorem	69
§5.5. The Riemann Hypothesis and the Prime Number Theorem	70
§5.6. Useful estimates involving primes	71
§5.7. Elementary proofs of the Prime Number Theorem	72
Problems on Chapter 5	72
Chapter 6. The Global Behavior of Arithmetic Functions	75
§6.1. Dirichlet series and arithmetic functions	75
§6.2. The uniqueness of representation of a Dirichlet series	77

---

§6.3. Multiplicative functions	79
§6.4. Generating functions and Dirichlet products	81
§6.5. Wintner's theorem	82
§6.6. Additive functions	85
§6.7. The average orders of $\omega(n)$ and $\Omega(n)$	86
§6.8. The average order of an additive function	87
§6.9. The Erdős-Wintner theorem	88
Problems on Chapter 6	89
Chapter 7. The Local Behavior of Arithmetic Functions	93
§7.1. The normal order of an arithmetic function	93
§7.2. The Turán-Kubilius inequality	94
§7.3. Maximal order of the divisor function	99
§7.4. An upper bound for $d(n)$	101
§7.5. Asymptotic densities	103
§7.6. Perfect numbers	106
§7.7. Sierpiński, Riesel, and Romanov	106
§7.8. Some open problems of an elementary nature	108
Problems on Chapter 7	109
Chapter 8. The Fascinating Euler Function	115
§8.1. The Euler function	115
§8.2. Elementary properties of the Euler function	117
§8.3. The average order of the Euler function	118
§8.4. When is $\phi(n)\sigma(n)$ a square?	119
§8.5. The distribution of the values of $\phi(n)/n$	121
§8.6. The local behavior of the Euler function	122
Problems on Chapter 8	124
Chapter 9. Smooth Numbers	127
§9.1. Notation	127
§9.2. The smallest prime factor of an integer	127
§9.3. The largest prime factor of an integer	131
§9.4. The Rankin method	137
§9.5. An application to pseudoprimes	141
§9.6. The geometric method	145
§9.7. The best known estimates on $\Psi(x, y)$	146

§9.8. The Dickman function	147
§9.9. Consecutive smooth numbers	149
Problems on Chapter 9	150
Chapter 10. The Hardy-Ramanujan and Landau Theorems	157
§10.1. The Hardy-Ramanujan inequality	157
§10.2. Landau's theorem	159
Problems on Chapter 10	164
Chapter 11. The <i>abc</i> Conjecture and Some of Its Applications	167
§11.1. The <i>abc</i> conjecture	167
§11.2. The relevance of the condition $\varepsilon > 0$	168
§11.3. The Generalized Fermat Equation	171
§11.4. Consecutive powerful numbers	172
§11.5. Sums of $k$ -powerful numbers	172
§11.6. The Erdős-Woods conjecture	173
§11.7. A problem of Gandhi	174
§11.8. The $k$ - <i>abc</i> conjecture	175
Problems on Chapter 11	176
Chapter 12. Sieve Methods	179
§12.1. The sieve of Eratosthenes	179
§12.2. The Brun sieve	180
§12.3. Twin primes	184
§12.4. The Brun combinatorial sieve	187
§12.5. A Chebyshev type estimate	187
§12.6. The Brun-Titchmarsh theorem	188
§12.7. Twin primes revisited	190
§12.8. Smooth shifted primes	191
§12.9. The Goldbach conjecture	192
§12.10. The Schnirelman theorem	194
§12.11. The Selberg sieve	198
§12.12. The Brun-Titchmarsh theorem from the Selberg sieve	201
§12.13. The Large sieve	202
§12.14. Quasi-squares	203
§12.15. The smallest quadratic nonresidue modulo $p$	204
Problems on Chapter 12	206

---

Chapter 13. Prime Numbers in Arithmetic Progression	217
§13.1. Quadratic residues	217
§13.2. The proof of the Quadratic Reciprocity Law	220
§13.3. Primes in arithmetic progressions with small moduli	222
§13.4. The Primitive Divisor theorem	224
§13.5. Comments on the Primitive Divisor theorem	227
Problems on Chapter 13	228
Chapter 14. Characters and the Dirichlet Theorem	233
§14.1. Primitive roots	233
§14.2. Characters	235
§14.3. Theorems about characters	236
§14.4. $L$ -series	240
§14.5. $L(1, \chi)$ is finite if $\chi$ is a non-principal character	242
§14.6. The nonvanishing of $L(1, \chi)$ : first step	243
§14.7. The completion of the proof of the Dirichlet theorem	244
Problems on Chapter 14	247
Chapter 15. Selected Applications of Primes in Arithmetic Progression	251
§15.1. Known results about primes in arithmetical progressions	251
§15.2. Some Diophantine applications	254
§15.3. Primes $p$ with $p - 1$ squarefree	257
§15.4. More applications of primes in arithmetic progressions	259
§15.5. Probabilistic applications	261
Problems on Chapter 15	263
Chapter 16. The Index of Composition of an Integer	267
§16.1. Introduction	267
§16.2. Elementary results	268
§16.3. Mean values of $\lambda$ and $1/\lambda$	270
§16.4. Local behavior of $\lambda(n)$	273
§16.5. Distribution function of $\lambda(n)$	275
§16.6. Probabilistic results	276
Problems on Chapter 16	279
Appendix: Basic Complex Analysis Theory	281
§17.1. Basic definitions	281

---

§17.2. Infinite products	283
§17.3. The derivative of a function of a complex variable	284
§17.4. The integral of a function along a path	285
§17.5. The Cauchy theorem	287
§17.6. The Cauchy integral formula	289
Solutions to Even-Numbered Problems	291
Solutions to problems from Chapter 1	291
Solutions to problems from Chapter 2	295
Solutions to problems from Chapter 3	303
Solutions to problems from Chapter 4	309
Solutions to problems from Chapter 5	312
Solutions to problems from Chapter 6	318
Solutions to problems from Chapter 7	321
Solutions to problems from Chapter 8	334
Solutions to problems from Chapter 9	338
Solutions to problems from Chapter 10	351
Solutions to problems from Chapter 11	353
Solutions to problems from Chapter 12	356
Solutions to problems from Chapter 13	377
Solutions to problems from Chapter 14	384
Solutions to problems from Chapter 15	392
Solutions to problems from Chapter 16	401
Bibliography	405
Index	413