

---

# Contents

Preface	xii
Notation	xiii
Acknowledgments	xiv

## Part 1. Expansion in Cayley Graphs

Chapter 1. Expander graphs: Basic theory	3
§1.1. Expander graphs	4
§1.2. Connection with edge expansion	9
§1.3. Random walks on expanders	15
§1.4. Random graphs as expanders	17
Chapter 2. Expansion in Cayley graphs, and Kazhdan's property (T)	23
§2.1. Kazhdan's property (T)	27
§2.2. Induced representations and property (T)	37
§2.3. The special linear group and property (T)	47
§2.4. A more elementary approach	55
Chapter 3. Quasirandom groups	57
§3.1. Mixing in quasirandom groups	62
§3.2. An algebraic description of quasirandomness	67
§3.3. A weak form of Selberg's 3/16 theorem	67
Chapter 4. The Balog-Szemerédi-Gowers lemma, and the Bourgain-Gamburd expansion machine	85
§4.1. The Balog-Szemerédi-Gowers lemma	87

---

§4.2. The Bourgain-Gamburd expansion machine	97
Chapter 5. Product theorems, pivot arguments, and the Larsen-Pink nonconcentration inequality	101
§5.1. The sum-product theorem	104
§5.2. Finite subgroups of $\mathrm{SL}_2$	110
§5.3. The product theorem in $\mathrm{SL}_2(k)$	120
§5.4. The product theorem in $\mathrm{SL}_d(k)$	125
§5.5. Proof of the Larsen-Pink inequality	129
Chapter 6. Nonconcentration in subgroups	135
§6.1. Expansion in thin subgroups	137
§6.2. Random generators expand	140
Chapter 7. Sieving and expanders	143
§7.1. Combinatorial sieving	146
§7.2. The strong approximation property	156
§7.3. Sieving in thin groups	160
<b>Part 2. Related Articles</b>	
Chapter 8. Cayley graphs and the algebra of groups	167
§8.1. A Hall-Witt identity for 2-cocycles	177
Chapter 9. The Lang-Weil bound	187
§9.1. The Stepanov-Bombieri proof of the Hasse-Weil bound	194
§9.2. The proof of the Lang-Weil bound	198
§9.3. Lang-Weil with parameters	200
Chapter 10. The spectral theorem and its converses for unbounded self-adjoint operators	203
§10.1. Self-adjointness and resolvents	207
§10.2. Self-adjointness and spectral measure	212
§10.3. Self-adjointness and flows	218
§10.4. Essential self-adjointness of the Laplace-Beltrami operator	224
Chapter 11. Notes on Lie algebras	227
§11.1. Abelian representations	233
§11.2. Engel's theorem and Lie's theorem	235
§11.3. Characterising semisimplicity	237
§11.4. Cartan subalgebras	242

§11.5. $\mathfrak{sl}_2$ representations	245
§11.6. Root spaces	247
§11.7. Classification of root systems	251
§11.8. Chevalley bases	258
§11.9. Casimirs and complete reducibility	263
Chapter 12. Notes on groups of Lie type	267
§12.1. Simple Lie groups over $\mathbf{C}$	268
§12.2. Chevalley groups	278
§12.3. Finite simple groups of Lie type	288
Bibliography	293
Index	301