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# 1

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## INTRODUCTION

Optics, as a field of science, is in its third millennium of life; yet in spite of its age, it remains remarkably vigorous and youthful. During the middle of the twentieth century, various events and discoveries gave new life, energy, and richness to the field. Especially important in this regard were (i) the introduction of the concepts and tools of Fourier analysis and communications theory into optics, primarily in the late 1940s and throughout the 1950s; (ii) the invention of the laser in late 1950s and its commercialization starting in the early 1960s; (iii) the origin of the field of nonlinear optics in the 1960s; (iv) the invention of the low-loss optical fiber in the early 1970s and the revolution in optical communications that followed; and (v) the rise of the young fields of nanophotonics and biophotonics. It is the thesis of this book that in parallel with these many advances, another important change has also taken place gradually but with an accelerating pace, namely, the infusion of statistical concepts and methods of analysis into the field of optics. It is to the role of such concepts in optics that this book is devoted.

The field we shall call “statistical optics” has a considerable history of its own. Many fundamental statistical problems were solved in the late nineteenth century and applied to acoustics and optics by Lord Rayleigh. The need for statistical methods in optics increased dramatically with the discovery of the quantized nature of light and, particularly, with the statistical interpretation of quantum mechanics introduced by Max Born. The introduction by E. Wolf in 1954 of an elegant and broad framework for considering the coherence properties of waves laid a foundation upon which many of the important statistical problems in optics could be treated in a unified