Contents

Preface vii

1 What Is Ergodic Theory? 1
Exercises 6

2 Topological Dynamical Systems 9
2.1 Basic Examples 10
2.2 Basic Constructions 15
2.3 Topological Transitivity 22
2.4 Transitivity of Subshifts 26
Exercises 29

3 Minimality and Recurrence 33
3.1 Minimality 34
3.2 Topological Recurrence 36
3.3 Recurrence in Extensions 39
Exercises 42

4 The C^*-Algebra C(K) and the Koopman Operator 45
4.1 Continuous Functions on Compact Spaces 46
4.2 The Space C(K) as a Commutative C^*-Algebra 51
4.3 The Koopman Operator 54
4.4 The Gelfand–Naimark Theorem 59
Supplement: Proof of the Gelfand–Naimark Theorem 60
Exercises 68

5 Measure-Preserving Systems 71
5.1 Examples 73
5.2 Measures on Compact Spaces 79
5.3 Haar Measures and Rotations 84
Supplement: Haar Measures on Homogeneous Spaces 87
Exercises 90

6 Recurrence and Ergodicity 95
6.1 The Measure Algebra and Invertible Systems 95
6.2 Recurrence 99
6.3 Ergodicity 104
Supplement: Induced Transformation and the Kakutani–Rokhlin Lemma 107
Exercises 110

7 The Banach Lattice L^p and the Koopman Operator 115
7.1 Banach Lattices and Positive Operators 117
7.2 The Space $L^p(X)$ as a Banach Lattice 122
7.3 The Koopman Operator and Ergodicity 124
Supplement: Interplay Between Lattice and Algebra Structure 129
Exercises 132

8 The Mean Ergodic Theorem 135
8.1 Von Neumann's Mean Ergodic Theorem 136
8.2 The Fixed Space and Ergodicity 140
8.3 Perron's Theorem and the Ergodicity of Markov Shifts 143
8.4 Mean Ergodic Operators 145
8.5 Operations with Mean Ergodic Operators 150
Supplement: Mean Ergodic Operator Semigroups 152
Exercises 158

9 Mixing Dynamical Systems 161
9.1 Strong Mixing 162
9.2 Weak Mixing 172
9.3 More Characterizations of Weakly Mixing Systems 179
9.4 Weak Mixing of All Orders 182
Exercises 187

10 Mean Ergodic Operators on $C(K)$ 191
10.1 Invariant Measures 192
10.2 Uniquely and Strictly Ergodic Systems 194
10.3 Mean Ergodicity of Group Rotations 197
10.4 Furstenberg's Theorem on Group Extensions 199
10.5 Application: Equidistribution 203
Exercises 206

11 The Pointwise Ergodic Theorem 211
11.1 Pointwise Ergodic Operators 212
11.2 Banach's Principle and Maximal Inequalities 214
11.3 Applications 218
Exercises 222

12 Isomorphisms and Topological Models 225
12.1 Point Isomorphisms and Factor Maps 226
12.2 Algebra Isomorphisms of Measure-Preserving Systems 230
12.3 Abstract Systems and Topological Models 235

12.4 The Stone Representation 240
12.5 Mean Ergodicity on Subalgebras 242
Exercises 245

13 Markov Operators 249
13.1 Examples and Basic Properties 250
13.2 Embeddings, Factor Maps and Isomorphisms 254
13.3 Markov Projections 257
13.4 Factors and Topological Models 262
13.5 Inductive Limits and the Invertible Extension 265
Exercises 271

14 Compact Groups 273
14.1 Compact Groups and the Haar Measure 273
14.2 The Character Group 276
14.3 The Pontryagin Duality Theorem 279
14.4 Monothetic Groups and Ergodic Rotations 283
Supplement: Characters of Discrete Abelian Groups 287
Exercises 288

15 Group Actions and Representations 291
15.1 Continuous Representations on Banach Spaces 292
15.2 Unitary Representations 297
15.3 Compact Group Actions 303
15.4 Markov Representations 306
Supplement: Abstract Compact Group Extensions 308
Exercises 312

16 The Jacobs–de Leeuw–Glicksberg Decomposition 317
16.1 Compact Semigroups 317
16.2 Weakly Compact Operator Semigroups 323
16.3 The Jacobs–de Leeuw–Glicksberg Decomposition 329
16.4 Cyclic Semigroups of Operators 336
Exercises 341

17 The Kronecker Factor and Systems with Discrete Spectrum 345
17.1 Semigroups of Markov Operators and the Kronecker Factor 345
17.2 Dynamical Systems with Discrete Spectrum 349
17.3 Disjointness of Weak Mixing and Discrete Spectrum 353
17.4 Examples 355
Exercises 363

18 The Spectral Theorem and Dynamical Systems 367
18.1 The Spectral Theorem 367
18.2 Spectral Decompositions and the Maximal Spectral Type 376
18.3 Discrete Measures and Eigenvalues 381
18.4 Dynamical Systems 386
Exercises 398

19 Topological Dynamics and Colorings 405
19.1 The Stone–Čech Compactification 406
19.2 The Semigroup Structure on βS 409
19.3 Topological Dynamics Revisited 412
19.4 Hindman's Theorem 416
19.5 From Coloring to Recurrence Results 419
19.6 From Recurrence to Coloring Results 424
Exercises 429

20 Arithmetic Progressions and Ergodic Theory 433
20.1 From Ergodic Theory to Arithmetic Progressions 434
20.2 Back from Arithmetic Progressions to Ergodic Theory 437
20.3 The Host–Kra Theorem 444
20.4 Furstenberg's Multiple Recurrence Theorem 451
20.5 The Furstenberg–Sárközy Theorem 455
Exercises 457

21 More Ergodic Theorems 461
21.1 Weighted Ergodic Theorems 461
21.2 Wiener–Wintner and Return Time Theorems 464
21.3 Linear Sequences as Good Weights 469
21.4 Subsequential Ergodic Theorems 470
21.5 Even More Ergodic Theorems 474
Exercises 476

A Topology 479

B Measure and Integration Theory 491

C Functional Analysis 507

D Operator Theory on Hilbert Spaces 521

E The Riesz Representation Theorem 543

F Standard Probability Spaces 557

G Theorems of Eberlein, Grothendieck, and Ellis 563

Bibliography 577

Index 613

Symbol Index 625