

Diffusions, Markov Processes, and Martingales

Volume 2: ITÔ CALCULUS

2nd Edition

L. C. G. ROGERS

*School of Mathematical Sciences,
University of Bath*

and

DAVID WILLIAMS

*Department of Mathematics,
University of Wales, Swansea*



CAMBRIDGE
UNIVERSITY PRESS

Contents

Some Frequently Used Notation

xiv

CHAPTER IV. INTRODUCTION TO ITÔ CALCULUS

TERMINOLOGY AND CONVENTIONS

R-processes and *L*-processes

Usual conditions, etc.

Important convention about time 0

1. SOME MOTIVATING REMARKS

1. Itô integrals	2
2. Integration by parts	4
3. Itô's formula for Brownian motion	8
4. A rough plan of the chapter	9

2. SOME FUNDAMENTAL IDEAS: PREVISIBLE PROCESSES, LOCALIZATION, etc.

Predictable processes

5. Basic integrands $Z(S, T]$	10
6. Predictable processes on $(0, \infty)$, \mathcal{F} , $b\mathcal{F}$, $b\mathcal{E}$	11
<i>Finite-variation and integrable-variation processes</i>	
7. FV_0 and IV_0 processes	14
8. Preservation of the martingale property	14

Localization

9. $H(0, T], X^T$	15
10. Localization of integrands, $lb\mathcal{F}$	16
11. Localization of integrators, $\mathcal{M}_{0,loc}$, $FV\mathcal{M}_{0,loc}$ etc.	17
12. Nil desperandum!	18
13. Extending stochastic integrals by localization	20
14. Local martingales, \mathcal{M}_{loc} , and the Fatou lemma	21
<i>Semimartingales as integrators</i>	
15. Semimartingales, \mathcal{S}	23
16. Integrators	24
<i>Likelihood ratios</i>	
17. Martingale property under change of measure	25

3. THE ELEMENTARY THEORY OF FINITE-VARIATION PROCESSES	
18. Itô's formula for FV functions	27
19. The Doléans exponential $\mathcal{E}(x)$	29
<i>Applications to Markov chains with finite state-space</i>	
20. Martingale problems	30
21. Probabilistic interpretation of \mathcal{Q}	33
22. Likelihood ratios and some key distributions	37
4. STOCHASTIC INTEGRALS: THE L^2 THEORY	
23. Orientation	42
24. Stable spaces of \mathcal{M}_0^2 , $c\mathcal{M}_0^2$, $d\mathcal{M}_0^2$	42
25. Elementary stochastic integrals relative to M in \mathcal{M}_0^2	45
26. The processes $[M]$ and $[M, N]$	46
27. Constructing stochastic integrals in L^2	47
28. The Kunita–Watanabe inequalities	50
5. STOCHASTIC INTEGRALS WITH RESPECT TO CONTINUOUS SEMIMARTINGALES	
29. Orientation	52
30. Quadratic variation for continuous local martingales	52
31. Canonical decomposition of a continuous semimartingale	57
32. Itô's formula for continuous semimartingales	58
6. APPLICATIONS OF ITÔ'S FORMULA	
33. Lévy's theorem	63
34. Continuous local martingales as time-changes of Brownian motion	64
35. Bessel processes; skew products; etc.	69
36. Brownian martingale representation	73
37. Exponential semimartingales; estimates	75
38. Cameron–Martin–Girsanov change of measure	79
39. First applications: Doob h -transforms; hitting of spheres; etc.	83
40. Further applications: bridges; excursions; etc.	86
41. Explicit Brownian martingale representation	89
42. Burkholder–Davis–Gundy inequalities	93
43. Semimartingale local time; Tanaka's formula	95
44. Study of joint continuity	99
45. Local time as an occupation density; generalized Itô–Tanaka formula	102
46. The Stratonovich calculus	106
47. Riemann-sum approximation to Itô and Stratonovich integrals; simulation	108

CHAPTER V. STOCHASTIC DIFFERENTIAL EQUATIONS AND DIFFUSIONS

1. INTRODUCTION

1. What is a diffusion in \mathbb{R}^n ?	110
2. FD diffusions recalled	112
3. SDEs as a means of constructing diffusions	113
4. Example: Brownian motion on a surface	114
5. Examples: modelling noise in physical systems	114
6. Example: Skorokhod's equation	117
7. Examples: control problems	119

2. PATHWISE UNIQUENESS, STRONG SDEs, AND FLOWS

8. Our general SDE; previsible path functionals; diffusion SDEs	122
9. Pathwise uniqueness; exact SDEs	124
10. Relationship between exact SDEs and strong solutions	125
11. The Itô existence and uniqueness result	128
12. Locally Lipschitz SDEs; Lipschitz properties of $a^{1/2}$	132
13. Flows; the diffeomorphism theorem; time-reversed flows	136
14. Carverhill's noisy North–South flow on a circle	141
15. The martingale optimality principle in control	144

3. WEAK SOLUTIONS, UNIQUENESS IN LAW

16. Weak solutions of SDEs; Tanaka's SDE	149
17. 'Exact equals weak plus pathwise unique'	151
18. Tsirel'son's example	155

4. MARTINGALE PROBLEMS, MARKOV PROPERTY

19. Definition; orientation	158
20. Equivalence of the martingale-problem and 'weak' formulations	160
21. Martingale problems and the strong Markov property	162
22. Appraisal and consolidation: where we have reached	163
23. Existence of solutions to the martingale problem	166
24. The Stroock–Varadhan uniqueness theorem	170
25. Martingale representation	173
<i>Transformation of SDEs</i>	
26. Change of time scale; Girsanov's SDE.	175
27. Change of measure	177
28. Change of state–space; scale; Zvonkin's observation; the Doss–Sussmann method.	178
29. Krylov's example	181

5. OVERTURE TO STOCHASTIC DIFFERENTIAL GEOMETRY

30. Introduction; some key ideas; Stratonovich-to-Itô conversion	182
31. Brownian motion on a submanifold of \mathbb{R}^N	186

32. Parallel displacement; Riemannian connections	193
33. Extrinsic theory of $\text{BM}^{\text{hor}}(\mathcal{O}(\Sigma))$; rolling without slipping; martin- gales on manifolds; etc.	198
34. Intrinsic theory; normal coordinates; structural equations; diffu- sions on manifolds; etc.(!)	203
35. Brownian motion on Lie groups.	224
36. Dynkin's Brownian motion of ellipses; hyperbolic space interpret- ation; etc.	239
37. Khasminskii's method for studying stability; random vibrations.	246
38. Hörmander's theorem; Malliavin calculus; stochastic pullback; curvature.	250
6. ONE-DIMENSIONAL SDEs	
39. A local-time criterion for pathwise uniqueness	263
40. The Yamada–Watanabe pathwise uniqueness theorem	265
41. The Nakao pathwise-uniqueness theorem	266
42. Solution of a variance control problem	267
43. A comparison theorem	269
7. ONE-DIMENSIONAL DIFFUSIONS	
44. Orientation	270
45. Regular diffusions	271
46. The scale function, s	273
47. The speed measure, m ; time substitution	276
48. Example: the Bessel SDE	284
49. Diffusion local time	289
50. Analytical aspects	291
51. Classification of boundary points	295
52. Khasminskii's test for explosion	297
53. An ergodic theorem for 1-dimensional diffusions	300
54. Coupling of 1-dimensional diffusions	301

CHAPTER VI. THE GENERAL THEORY

1. ORIENTATION	
1. Preparatory remarks	304
2. Lévy processes	308
2. DEBUT AND SECTION THEOREMS	
3. Progressive processes	313
4. Optional processes, \mathcal{O} ; optional times.	315
5. The 'optional' section theorem	317
6. Warning (not to be skipped)	318

3. OPTIONAL PROJECTIONS AND FILTERING	
7. Optional projection ${}^{\circ}X$ of X	319
8. The innovations approach to filtering	322
9. The Kalman–Bucy filter	327
10. The Bayesian approach to filtering; a change-detection filter	329
11. Robust filtering	331
4. CHARACTERIZING PREVISIBLE TIMES	
12. Previsible stopping times; PFA theorem	332
13. Totally inaccessible and accessible stopping times.	334
14. Some examples.	336
15. Meyer’s previsibility theorem for Markov processes	338
16. Proof of the PFA theorem.	340
17. The σ -algebras $\mathcal{F}(\rho-)$, $\mathcal{F}(\rho)$, $\mathcal{F}(\rho+)$	343
18. Quasi-left-continuous filtrations	346
5. DUAL PREVISIBLE PROJECTIONS	
19. The previsible section theorem; the previsible projection ${}^{\circ}X$ of X	347
20. Doléans’ characterization of FV processes	349
21. Dual previsible projections, compensators	350
22. Cumulative risk	352
23. Some Brownian motion examples	354
24. Decomposition of a continuous semimartingale	358
25. Proof of the basic (μ, A) correspondence	359
26. Proof of the Doléans ‘optional’ characterization result	360
27. Proof of the Doléans ‘previsible’ characterization result	361
28. Lévy systems for Markov processes	364
6. THE MEYER DECOMPOSITION THEOREM	
29. Introduction.	367
30. The Doléans proof of the Meyer decomposition	369
31. Regular class (D) submartingales; approximation to compensators	372
32. The local form of the decomposition theorem	374
33. An L^2 bounded local martingale which is not a martingale	375
34. The $\langle M \rangle$ process	376
35. Last exits and equilibrium charge	377
7. STOCHASTIC INTEGRATION: THE GENERAL CASE	
36. The quadratic variation process $[M]$	382
37. Stochastic integrals with respect to local martingales	388
38. Stochastic integrals with respect to semimartingales	391
39. Itô’s formula for semimartingales	394
40. Special semimartingales.	394
41. Quasimartingales	396

8. ITÔ EXCURSION THEORY	
42. Introduction.	398
43. Excursion theory for a finite Markov chain.	400
44. Taking stock	405
45. Local time L at a regular extremal point a	406
46. Some technical points: hypothèses droites, etc.	410
47. The Poisson point process of excursions	413
48. Markovian character of n	416
49. Marking the excursions.	418
50. Last-exit decomposition; calculation of the excursion law n	420
51. The Skorokhod embedding theorem	425
52. Diffusion properties of local time in the space variable; the Ray–Knight theorem.	428
53. Arcsine law for Brownian motion	431
54. Resolvent density of a 1-dimensional diffusion.	432
55. Path decomposition of Brownian motions and of excursions.	433
56. An illustrative calculation	438
57. Feller Brownian motions	439
58. Example: censoring and reweighting of excursion laws	442
59. Excursion theory by stochastic calculus: McGill’s lemma	445
REFERENCES	449
INDEX	469