

Paul Frank Baum · Guillermo Cortiñas  
Ralf Meyer · Rubén Sánchez-García  
Marco Schlichting · Bertrand Toën

# Topics in Algebraic and Topological K-Theory

Editor: Guillermo Cortiñas

 Springer

---

# Contents

## ***K*-Theory for Group $C^*$ -algebras**

<i>Paul F. Baum, Rubén J. Sánchez-García</i> .....	1
1 Introduction .....	1
2 $C^*$ -algebras .....	2
2.1 Definitions .....	2
2.2 Examples .....	3
2.3 The Reduced $C^*$ -algebra of a Group .....	4
2.4 Two Classical Theorems .....	6
2.5 The Categorical Viewpoint .....	6
3 <i>K</i> -Theory of $C^*$ -algebras .....	7
3.1 Definition for Unital $C^*$ -algebras .....	7
3.2 Bott Periodicity .....	8
3.3 Definition for Non-Unital $C^*$ -algebras .....	8
3.4 Functoriality .....	9
3.5 More on Bott Periodicity .....	9
3.6 Topological <i>K</i> -Theory .....	10
4 Proper $G$ -Spaces .....	11
5 Classifying Space for Proper Actions .....	12
5.1 Examples .....	13
6 Equivariant <i>K</i> -Homology .....	14
6.1 Definitions .....	14
6.2 Functoriality .....	18
6.3 The Index Map .....	18
7 The Discrete Case .....	19
7.1 Equivariant <i>K</i> -Homology .....	19
7.2 Some Results on Discrete Groups .....	20
7.3 Corollaries of the Baum–Connes Conjecture .....	21
8 The Compact Case .....	21
9 Equivariant <i>K</i> -Homology for $G$ - $C^*$ -algebras .....	22
10 The Conjecture with Coefficients .....	24

11	Hilbert Modules	25
11.1	Definitions and Examples	26
11.2	The Reduced Crossed-Product $C_r^*(G, A)$	28
11.3	Push-Forward of Hilbert Modules	30
12	Homotopy Made Precise and $KK$ -Theory	30
13	Equivariant $KK$ -Theory	33
14	The Index Map	35
14.1	The Kasparov Product	35
14.2	The Kasparov Descent Map	35
14.3	Definition of the Index Map	36
14.4	The Index Map with Coefficients	36
15	A Brief History of $K$ -Theory	37
15.1	The $K$ -Theory Genealogy Tree	37
15.2	The Hirzebruch–Riemann–Roch Theorem	38
15.3	The Unity of $K$ -Theory	39
	References	40

### Universal Coefficient Theorems and Assembly Maps in $KK$ -Theory

<i>Ralf Meyer</i>		45
1	Introduction	45
2	Kasparov Theory and Baum–Connes Conjecture	46
2.1	Kasparov Theory via Its Universal Property	46
2.2	Subcategories in $\mathfrak{K}\mathfrak{R}^G$	60
3	Homological Algebra	65
3.1	Homological Ideals in Triangulated Categories	67
3.2	From Homological Ideals to Derived Functors	72
3.3	Universal Coefficient Theorems	93
	References	100

### Algebraic v. Topological $K$ -Theory: A Friendly Match

<i>Guillermo Cortiñas</i>		103
1	Introduction	103
2	The Groups $K_n$ for $n \leq 1$	104
2.1	Definition and Basic Properties of $K_j$ for $j = 0, 1$	104
2.2	Matrix-Stable Functors	110
2.3	Sum Rings and Infinite Sum Rings	112
2.4	The Excision Sequence for $K_0$ and $K_1$	113
2.5	Negative $K$ -Theory	116
3	Topological $K$ -Theory	117
3.1	Topological $K$ -Theory of Banach Algebras	118
3.2	Bott Periodicity	121
4	Polynomial Homotopy and Karoubi–Villamayor $K$ -Theory	123
5	Homotopy $K$ -Theory	126
5.1	Definition and Basic Properties of $KH$	126
5.2	$KH$ for $K_0$ -Regular Rings	128
5.3	Toeplitz Ring and the Fundamental Theorem for $KH$	129

6	Quillen's Higher $K$ -Theory . . . . .	131
6.1	Classifying Spaces . . . . .	131
6.2	Perfect Groups and the Plus Construction for $BG$ . . . . .	132
6.3	Functoriality Issues . . . . .	136
6.4	Relative $K$ -Groups and Excision . . . . .	136
6.5	Locally Convex Algebras . . . . .	138
6.6	Fréchet $m$ -Algebras with Approximate Units . . . . .	139
6.7	Fundamental Theorem and the Toeplitz Ring . . . . .	140
7	Comparison Between Algebraic and Topological $K$ -Theory I . . . . .	142
7.1	Stable $C^*$ -Algebras . . . . .	142
7.2	Stable Banach Algebras . . . . .	143
8	Topological $K$ -Theory for Locally Convex Algebras . . . . .	144
8.1	Diffeotopy $KV$ . . . . .	144
8.2	Diffeotopy $K$ -Theory . . . . .	146
8.3	Bott Periodicity . . . . .	147
9	Comparison Between Algebraic and Topological $K$ -Theory II . . . . .	149
9.1	The Diffeotopy Invariance Theorem . . . . .	149
9.2	$KH$ of Stable Locally Convex Algebras . . . . .	151
10	$K$ -Theory Spectra . . . . .	152
10.1	Quillen's $K$ -Theory Spectrum . . . . .	152
10.2	$KV$ -Theory Spaces . . . . .	153
10.3	The Homotopy $K$ -Theory Spectrum . . . . .	155
11	Primary and Secondary Chern Characters . . . . .	156
11.1	Cyclic Homology . . . . .	156
11.2	Primary Chern Character and Infinitesimal $K$ -Theory . . . . .	157
11.3	Secondary Chern Characters . . . . .	158
11.4	Application to $KD$ . . . . .	160
12	Comparison Between Algebraic and Topological $K$ -Theory III . . . . .	160
12.1	Stable Fréchet Algebras . . . . .	160
12.2	Stable Locally Convex Algebras: The Comparison Sequence . . . . .	161
	References . . . . .	163

**Higher Algebraic  $K$ -Theory** (After Quillen, Thomason and Others)

	<i>Marco Schlichting</i> . . . . .	167
1	Introduction . . . . .	167
2	The $K$ -Theory of Exact Categories . . . . .	168
2.1	The Grothendieck Group of an Exact Category . . . . .	168
2.2	Quillen's $Q$ -Construction and Higher $K$ -Theory . . . . .	172
2.3	Quillen's Fundamental Theorems . . . . .	177
2.4	Negative $K$ -Groups . . . . .	180
3	Algebraic $K$ -Theory and Triangulated Categories . . . . .	183
3.1	The Grothendieck-Group of a Triangulated Category . . . . .	183
3.2	The Thomason–Waldhausen Localization Theorem . . . . .	187
3.3	Quillen's Fundamental Theorems Revisited . . . . .	197

3.4	Thomason's Mayer-Vietoris Principle	201
3.5	Projective Bundle Theorem and Regular Blow-Ups	207
4	Beyond Triangulated Categories	211
4.1	Statement of Results	211
A	Appendix	215
A.1	Background from Topology	215
A.2	Background on Triangulated Categories	219
A.3	The Derived Category of Quasi-Coherent Sheaves	226
A.4	Proof of Compact Generation of $D_Z \text{Qcoh}(X)$	230
	References	237

### Lectures on DG-Categories

	<i>Toën Bertrand</i>	243
1	Introduction	243
2	Lecture 1: DG-Categories and Localization	244
2.1	The Gabriel–Zisman Localization	244
2.2	Bad Behavior of the Gabriel–Zisman Localization	247
2.3	DG-Categories and DG-Functors	250
2.4	Localizations as a dg-Category	265
3	Lecture 2: Model Categories and dg-Categories	265
3.1	Reminders on Model Categories	266
3.2	Model Categories and dg-Categories	271
4	Lecture 3: Structure of the Homotopy Category of dg-Categories	275
4.1	Maps in the Homotopy Category of dg-Categories	276
4.2	Existence of Internal Homs	280
4.3	Existence of Localizations	281
4.4	Triangulated dg-Categories	284
5	Lecture 4: Some Applications	288
5.1	Functorial Cones	288
5.2	Some Invariants	291
5.3	Descent	293
5.4	Saturated dg-Categories and Secondary K-Theory	295
	References	301