

# Contents

<b>1</b>	<b>Preliminaries</b> .....	1
1.1	Notations .....	3
1.2	The $\Gamma$ -Function .....	5
1.3	Basic Results Related to the Sphere .....	6
<b>2</b>	<b>Spherical Harmonics</b> .....	11
2.1	Spherical Harmonics Through Primitive Spaces .....	11
2.1.1	Spaces of Homogeneous Polynomials .....	13
2.1.2	Legendre Harmonic and Legendre Polynomial .....	17
2.1.3	Spherical Harmonics .....	19
2.2	Addition Theorem and Its Consequences .....	20
2.3	A Projection Operator .....	26
2.4	Relations Among Polynomial Spaces .....	30
2.5	The Funk–Hecke Formula .....	34
2.6	Legendre Polynomials: Representation Formulas .....	36
2.6.1	Rodrigues Representation Formula .....	36
2.6.2	Integral Representation Formulas .....	39
2.7	Legendre Polynomials: Properties .....	41
2.7.1	Integrals, Orthogonality .....	42
2.7.2	Differential Equation and Distribution of Roots .....	43
2.7.3	Recursion Formulas .....	45
2.7.4	Generating Function .....	52
2.7.5	Values and Bounds .....	56
2.8	Completeness .....	60
2.8.1	Completeness in $C(\mathbb{S}^{d-1})$ .....	60
2.8.2	Completeness in $C(\mathbb{S}^{d-1})$ via the Poisson Identity ...	64
2.8.3	Convergence of Fourier–Laplace Series .....	66
2.8.4	Completeness in $L^2(\mathbb{S}^{d-1})$ .....	69
2.9	The Gegenbauer Polynomials .....	71

2.10	The Associated Legendre Functions .....	74
2.10.1	Definition and Representation Formulas .....	74
2.10.2	Properties .....	77
2.10.3	Normalized Associated Legendre Functions .....	80
2.11	Generating Orthonormalized Bases for Spherical Harmonic Spaces .....	81
<b>3</b>	<b>Differentiation and Integration over the Sphere .....</b>	<b>87</b>
3.1	The Laplace–Beltrami Operator .....	87
3.2	A Formula for the Laplace–Beltrami Operator .....	94
3.3	Spherical Harmonics As Eigenfunctions of the Laplace–Beltrami Operator .....	96
3.4	Some Integration Formulas .....	100
3.5	Some Differentiation Formulas .....	104
3.6	Some Integral Identities for Spherical Harmonics .....	106
3.7	Integral Identities Through the Funk–Hecke Formula .....	113
3.7.1	A Family of Integral Identities for Spherical Harmonics .....	114
3.7.2	Some Extensions .....	118
3.8	Sobolev Spaces on the Unit Sphere .....	119
3.9	Positive Definite Functions .....	124
<b>4</b>	<b>Approximation Theory .....</b>	<b>131</b>
4.1	Spherical Polynomials .....	132
4.2	Best Approximation on the Unit Sphere .....	135
4.2.1	The Approach to Best Approximation of Dai and Xu .....	136
4.2.2	The Approach to Best Approximation of Ragozin ...	142
4.2.3	Best Simultaneous Approximation Including Derivatives .....	149
4.2.4	Lebesgue Constants .....	150
4.2.5	Best Approximation for a Parameterized Family .....	152
4.3	Approximation on the Unit Disk .....	154
4.3.1	Orthogonal Polynomials .....	155
4.3.2	Properties of Orthogonal Polynomials over $\mathbb{B}^2$ .....	161
4.3.3	Orthogonal Expansions .....	162
<b>5</b>	<b>Numerical Quadrature .....</b>	<b>165</b>
5.1	The Use of Univariate Formulas .....	166
5.2	Composite Methods .....	172
5.2.1	The Centroid Method .....	174
5.2.2	General Composite Methods .....	176
5.2.3	Error Analysis .....	178
5.3	High Order Gauss-Type Methods .....	185
5.3.1	Efficiency of a High-Order Formula .....	188
5.3.2	The Centroid Method .....	189
5.3.3	An Alternative Approach .....	190

5.4	Integration of Scattered Data .....	193
5.5	Integration of Singular Functions .....	195
5.5.1	Singular Integrands .....	199
5.6	Quadrature over the Unit Disk .....	201
5.6.1	A Product Gauss Formula .....	202
5.7	Discrete Orthogonal Expansions .....	203
5.7.1	Hyperinterpolation over $\mathbb{S}^2$ .....	204
5.7.2	Hyperinterpolation over the Unit Disk .....	210
<b>6</b>	<b>Applications: Spectral Methods</b> .....	<b>211</b>
6.1	A Boundary Integral Equation .....	212
6.1.1	Convergence Theory .....	214
6.1.2	Quadrature .....	217
6.1.3	A Numerical Example .....	220
6.2	A Spectral Method for a Partial Differential Equation .....	224
6.2.1	Implementation .....	229
6.2.2	A Numerical Example .....	230
6.3	A Galerkin Method for a Beltrami-Type Equation .....	232
	<b>References</b> .....	<b>237</b>
	<b>Index</b> .....	<b>243</b>