The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems

Pavel Etingof and Frédéric Latour

OXFORD
UNIVERSITY PRESS

CONTENTS

1 Introduction 1

1.1 The quantum dynamical Yang–Baxter equation 1

1.1.1 The equation 1

1.1.2 Examples of solutions of QDYBE 1

1.1.3 The QDYBE with spectral parameter 2

1.1.4 Tensor category of representations 2

1.1.5 Gauge transformations and classification 3

1.1.6 Dynamical quantum groups 3

1.1.7 The classical dynamical Yang–Baxter equation 4

1.1.8 Examples of solutions of CDYBE 5

1.1.9 Classification of solutions for CDYBE 5

1.2 The fusion and exchange construction 6

1.2.1 Intertwining operators 6

1.2.2 The fusion and exchange operators 7

1.2.3 Fusion and exchange for quantum groups 7

1.2.4 The ABRR equation 8

1.2.5 The universal fusion operator 8

1.2.6 The dynamical twist equation 9

1.3 Traces of intertwiners and Macdonald functions 9

1.3.1 Trace functions 9

1.3.2 Commuting difference operators 10

1.3.3 Difference equations for the trace functions 10

1.3.4 Macdonald functions 10

1.3.5 Dynamical Weyl groups 12

2 Background material 15

2.1 Facts about $\mathfrak{sl}_2$ 15

2.2 Semisimple finite-dimensional Lie algebras and roots 16

2.3 Inner product on a simple Lie algebra 17

2.4 Chevalley generators 18

2.5 Representations of finite-dimensional semisimple Lie algebras 19

2.6 Irreducible highest weight modules; Shapovalov form 21

3 Intertwiners, fusion and exchange operators for Lie algebras 26

3.1 Intertwining operators 26

3.2 The fusion operator 27

3.3 The dynamical twist equation 28

3.4 The exchange operator 29

3.5 The ABRR equation 34

3.6 The universal fusion and exchange operators 38

4 Quantum groups 40
4.1 Hopf algebras 40
4.2 Representations of Hopf algebras 41
4.3 The quantum group $\mathfrak{U}_q(\mathfrak{sl}_2)$ 42
4.4 The quantum group $\mathfrak{U}_q(\mathfrak{g})$ 43
4.5 PBW for $\mathfrak{U}_q(\mathfrak{g})$ 44
4.6 The Hopf algebra structure on $\mathfrak{U}_q(\mathfrak{g})$ 45
4.7 Representation theory of $\mathfrak{U}_q(\mathfrak{g})$ 46
4.8 Formal version of quantum groups 47
4.9 Quasi-triangular Hopf algebras 48
4.10 Quasi-triangular Hopf algebras and representation theory 50
4.11 Quasi-triangularity and $\mathfrak{U}_q(\mathfrak{g})$ 53
4.12 Twisting 55
4.13 Quasi-classical limit for the QYBE 57
4.14 Quasi-classical limit for the QDYBE 58

5 Intertwiners, fusion and exchange operators for $\mathfrak{U}_q(\mathfrak{g})$ 61
5.1 Fusion operator for $\mathfrak{U}_q(\mathfrak{g})$ 61
5.2 Exchange operator for $\mathfrak{U}_q(\mathfrak{g})$ 63
5.3 The ABRR equation for $\mathfrak{U}_q(\mathfrak{g})$ 64
5.4 Quasi-classical limit for ABRR equation for $\mathfrak{U}_q(\mathfrak{g})$ 65

6 Dynamical R-matrices and integrable systems 70
6.1 Classical mechanics vs. quantum mechanics 70
6.2 Transfer matrix construction 71
6.3 Dynamical transfer matrix construction 72

7 Traces of intertwiners for $\mathfrak{U}_q(\mathfrak{g})$ 78
7.1 Generalized Macdonald- Ruijsenaars operators 78
7.2 Construction of $F_V(\lambda, \mu)$ 80
7.3 Quantum spin Calogero -Moser Hamiltonian 80
7.4 $F_V(\lambda, \mu)$ for $\mathfrak{sl}_2$ 85
7.5 Center of $\mathfrak{U}_q(\mathfrak{g})$ and quantum traces 88
7.6 The functions Z_V and X_V 92
7.7 The function $\tilde{G}$ 97
7.8 Macdonald -Ruijsenaars equations 107
7.9 Dual Macdonald -Ruijsenaars equations 107
7.10 The symmetry identity 112

8 Traces of intertwiners and Macdonald polynomials 114
8.1 Macdonald polynomials 114
8.2 Vector-valued characters 118

9 Dynamical Weyl group 127
9.1 Dynamical Weyl group (for $\mathfrak{g} = \mathfrak{sl}_2$) 127

9.2 Dynamical Weyl group (for any finite-dim. simple $\mathfrak{g}$) 131

References 135

Index 138