

Contents

Preface

vii

I. Introduction	1
1. Preliminaries and conventions	4
2. Premetrics and balls	6
3. Sequences	9
4. Cofiniteness	13
5. Quences	14
6. Almost inclusion	17
7. When premetrics and sequences do not suffice	19
7.1. Pointwise convergence	19
7.2. Riemann integrals	23
II. Families of sets	27
1. Isotone families of sets	27
2. Filters	29
2.1. Order	30
2.2. Free and principal filters	33
2.3. Sequential filters	35
2.4. Images, preimages, products	37
3. Grills	39
4. Duality between filters and grills	41
5. Triad: filters, filter-grills and ideals	42
6. Ultrafilters	43
7. Cardinality of the set of ultrafilters	46
8. Remarks on sequential filters	48
8.1. Countably based and Fréchet filters	48
8.2. Infima and products of filters	51
9. Contours and extensions	53

III. Convergences	55
1. Definitions and first examples	55
2. Preconvergences on finite sets	60
2.1. Preconvergences on two-point sets	60
2.2. Preconvergences on three-point sets	62
3. Induced (pre)convergence	64
4. Premetrizable convergences	65
5. Adherence and cover	67
6. Lattice of convergences	70
7. Finitely deep modification	71
8. Pointwise properties of convergence spaces	72
9. Convergences on a complete lattice	75
IV. Continuity	79
1. Continuous maps	79
2. Initial and final convergences	82
3. Initial and final convergences for multiple maps	86
4. Product convergence	89
4.1. Finite product	89
4.2. Infinite product	91
5. Functional convergences	92
6. Diagonal and product maps	94
6.1. Diagonal map	94
6.2. Product map	95
7. Initial and final convergences for product maps	96
8. Quotient	97
9. Convergence invariants	102
9.1. Premetrizability, metrizability	103
9.2. Isolated points, paving number, finite depth	104
9.3. Characters and weight	105
9.4. Density and separability	109
V. Pretopologies	115
1. Definition and basic properties	115
2. Principal adherences and inherences	121
3. Open and closed sets, closures, interiors, neighborhoods	128
4. Topologies	135
4.1. Topological modification	139

4.2. Induced topology	143
4.3. Product topology	144
5. Open maps and closed maps	147
6. Topological defect and sequential order	149
6.1. Iterated adherence and topological defect	149
6.2. Sequentially based convergence and sequential order	153
VI. Diagonality and regularity	161
1. More on contours	161
2. Diagonality	163
2.1. Various types of diagonality	165
2.2. Diagonal modification	170
3. Self-regularity	171
4. Topological regularity	176
5. Regularity with respect to another convergence	177
VII. Types of separation	179
1. Convergence separation	179
2. Regularity with respect to a family of sets	182
3. Functionally induced convergences	184
4. Real-valued functions	187
5. Functionally closed and open sets	188
6. Functional regularity (aka complete regularity)	191
7. Normality	197
8. Continuous extension of maps	205
9. Tietze's extension theorem	209
VIII. Pseudotopologies	213
1. Adherence, inherence	213
2. Pseudotopologies	216
3. Pseudotopologizer	218
4. Regularity and topologicity among pseudotopologies	221
5. Initial density in pseudotopologies	223
6. Natural convergence	225
7. Convergences on hyperspaces	226

IX. Compactness	231
1. Compact sets	231
2. Regularity and topologicity in compact spaces	238
3. Local compactness	240
4. Topologicity of hyperspace convergences	245
5. The Stone topology	247
6. Almost disjoint families	252
7. Compact families	256
8. Conditional compactness	261
8.1. Paratopologies	262
8.2. Countable compactness	262
8.3. Sequential compactness	265
9. Upper Kuratowski topology	271
10. More on covers	272
11. Cover-compactness	275
12. Pseudocompactness	279
X. Completeness in metric spaces	283
1. Complete metric spaces	283
2. Completely metrizable spaces	288
3. Metric spaces of continuous functions	290
4. Uniform continuity, extensions, and completion	292
XI. Completeness	297
1. Completeness with respect to a collection	297
2. Cocompleteness	299
3. Completeness number	302
4. Finitely complete convergences	305
5. Countably complete convergences	306
6. Preservation of completeness	307
7. Completeness of subspaces	309
8. Completeness of products	311
9. Conditionally complete convergences	314
10. Baire property	315
11. Strict completeness	317

XII. Connectedness	319
1. Connected spaces	319
2. Path connected and arc connected spaces	326
3. Components and quasi-components	328
4. Remarks on zero-dimensional spaces	333
XIII. Compactifications	335
1. Introduction	335
2. Compactifications of functionally regular topologies	338
3. Filters in lattices	343
4. Filters in lattices of closed and functionally closed sets	345
5. Maximality conditions	347
6. Čech-Stone compactification	349
XIV. Classification of spaces	355
1. Modifiers, projectors, and coprojectors	355
2. Functors, reflectors and coreflectors	360
3. Adherence-determined convergences	363
3.1. Reflective classes	364
3.2. Composable classes of filters	366
3.3. Conditional compactness	368
4. Convergences based in a class of filters	370
5. Other \mathbb{F}_0 -composable classes of filters	373
6. Functorial inequalities and classification of spaces	375
7. Reflective and coreflective hulls	380
8. Conditional compactness and cover-compactness	385
XV. Classification of maps	389
1. Various types of quotient maps	389
1.1. Remarks on the quotient convergence	389
1.2. Topologically quotient maps	390
1.3. Hereditarily quotient maps	393
1.4. Quotient maps relative to a reflector	395
1.5. Biquotient maps	396
1.6. Almost open maps	397
1.7. Countably biquotient map	398
2. Interactions between maps and spaces	398
3. Compact relations	400
4. Product of spaces and of maps	404

XVI. Spaces of maps	411
1. Evaluation and adjoint maps	412
2. Adjoint maps on spaces of continuous maps	415
3. Fundamental convergences on spaces of continuous maps	416
4. Pointwise convergence	417
5. Natural convergence	420
5.1. Continuity of limits	421
5.2. Exponential law	423
5.3. Finer subspaces and natural convergence	425
5.4. Continuity of adjoint maps	427
5.5. Initial structures for adjoint maps	429
6. Compact subsets of function spaces (Ascoli-Arzelà)	431
XVII. Duality	437
1. Natural duality	437
2. Modified duality	443
3. Concrete characterizations of bidual reflectors	449
4. Epitopologies	450
5. Functionally embedded convergences	451
6. Exponential hulls and exponential objects	453
7. Duality and product theorems	459
8. Non-Fréchet product of two Fréchet compact topologies	466
9. Spaces of real-valued continuous functions	469
9.1. Cauchy completeness	469
9.2. Completeness number	470
9.3. Character and weight	471
XVIII. Functional partitions and metrization	475
1. Introduction	475
2. Perfect normality	475
3. Pseudometrics	478
4. Functional covers and partitions	481
5. Paracompactness	488
6. Fragmentations of partitions of unity	491
7. Metrization theorems	494

A. Set theory	497
1. Axiomatic set theory	497
2. Basic set theory	499
3. Natural numbers	501
4. Cardinality	502
5. Continuum	507
6. Order	509
7. Lattice	510
8. Well ordered sets	512
9. Ordinal numbers	514
10. Ordinal arithmetic	517
11. Ordinal-cardinal numbers	519
<i>Bibliography</i>	523
<i>List of symbols</i>	529
<i>Index</i>	541