Contents

1. Introduction 1
1.1 Motivation 1
1.2 Background 2
1.2.1 Problem Specification and Geometry Preparation 2
1.2.2 Selection of Governing Equations and Boundary Conditions 3
1.2.3 Selection of Gridding Strategy and Numerical Method 3
1.2.4 Assessment and Interpretation of Results 4
1.3 Overview 4
1.4 Notation 4

2. Conservation Laws and the Model Equations 7
2.1 Conservation Laws 7
2.2 The Navier–Stokes and Euler Equations 8
2.3 The Linear Convection Equation 11
2.3.1 Differential Form 11
2.3.2 Solution in Wave Space 12
2.4 The Diffusion Equation 13
2.4.1 Differential Form 13
2.4.2 Solution in Wave Space 14
2.5 Linear Hyperbolic Systems 15
Exercises 17

3. Finite-Difference Approximations 19
3.1 Meshes and Finite-Difference Notation 19
3.2 Space Derivative Approximations 21
3.3 Finite-Difference Operators 22
3.3.1 Point Difference Operators 22
3.3.2 Matrix Difference Operators 23
3.3.3 Periodic Matrices 26
3.3.4 Circulant Matrices 27
3.4 Constructing Differencing Schemes of Any Order 28
3.4.1 Taylor Tables 28
3.4.2 Generalization of Difference Formulas 31

3.4.3 Lagrange and Hermite Interpolation Polynomials 33
3.4.4 Practical Application of Padé Formulas 35
3.4.5 Other Higher-Order Schemes 36
3.5 Fourier Error Analysis 37
3.5.1 Application to a Spatial Operator 37
3.6 Difference Operators at Boundaries 41
3.6.1 The Linear Convection Equation 41
3.6.2 The Diffusion Equation 44
Exercises 46

4. The Semi-Discrete Approach 49
4.1 Reduction of PDE's to ODE's 50
4.1.1 The Model ODE's 50
4.1.2 The Generic Matrix Form 51
4.2 Exact Solutions of Linear ODE's 51
4.2.1 Eigensystems of Semi-discrete Linear Forms 52
4.2.2 Single ODE's of First and Second Order 53
4.2.3 Coupled First-Order ODE's 54
4.2.4 General Solution of Coupled ODE's with Complete Eigensystems 56
4.3 Real Space and Eigenspace 58
4.3.1 Definition 58
4.3.2 Eigenvalue Spectrums for Model ODE's 59
4.3.3 Eigenvectors of the Model Equations 60
4.3.4 Solutions of the Model ODE's 62
4.4 The Representative Equation 64
Exercises 65

5. Finite-Volume Methods 67
5.1 Basic Concepts 67
5.2 Model Equations in Integral Form 69
5.2.1 The Linear Convection Equation 69
5.2.2 The Diffusion Equation 70
5.3 One-Dimensional Examples 70
5.3.1 A Second-Order Approximation to the Convection Equation 71
5.3.2 A Fourth-Order Approximation to the Convection Equation 72
5.3.3 A Second-Order Approximation to the Diffusion Equation 74
5.4 A Two-Dimensional Example 76
Exercises 79

6. Time-Marching Methods for ODE'S 81
6.1 Notation 82
6.2 Converting Time-Marching Methods to OΔE's 83
6.3 Solution of Linear OΔE's with Constant Coefficients 84
6.3.1 First- and Second-Order Difference Equations 84
6.3.2 Special Cases of Coupled First-Order Equations 86
6.4 Solution of the Representative OΔE's 87
6.4.1 The Operational Form and its Solution 87
6.4.2 Examples of Solutions to Time-Marching OΔE's 88
6.5 The λ–σ Relation 89
6.5.1 Establishing the Relation 89
6.5.2 The Principal σ-Root 90
6.5.3 Spurious σ-Roots 91
6.5.4 One-Root Time-Marching Methods 92
6.6 Accuracy Measures of Time-Marching Methods 92
6.6.1 Local and Global Error Measures 92
6.6.2 Local Accuracy of the Transient Solution $(er_\lambda, |\sigma|, er_\omega)$ 93
6.6.3 Local Accuracy of the Particular Solution (er_μ) 94
6.6.4 Time Accuracy for Nonlinear Applications 95
6.6.5 Global Accuracy 96
6.7 Linear Multistep Methods 96
6.7.1 The General Formulation 97
6.7.2 Examples 97
6.7.3 Two-Step Linear Multistep Methods 100
6.8 Predictor–Corrector Methods 101
6.9 Runge–Kutta Methods 103
6.10 Implementation of Implicit Methods 105
6.10.1 Application to Systems of Equations 105
6.10.2 Application to Nonlinear Equations 106
6.10.3 Local Linearization for Scalar Equations 107
6.10.4 Local Linearization for Coupled Sets of Nonlinear Equations 110
Exercises 112

7. Stability of Linear Systems 115
7.1 Dependence on the Eigensystem 115
7.2 Inherent Stability of ODE's 116
7.2.1 The Criterion 116
7.2.2 Complete Eigensystems 117
7.2.3 Defective Eigensystems 117
7.3 Numerical Stability of OΔE's 118
7.3.1 The Criterion 118
7.3.2 Complete Eigensystems 118
7.3.3 Defective Eigensystems 119
7.4 Time–Space Stability and Convergence of OΔE's 119

7.5 Numerical Stability Concepts in the Complex σ-Plane 121
7.5.1 σ-Root Traces Relative to the Unit Circle 121
7.5.2 Stability for Small Δt 126
7.6 Numerical Stability Concepts in the Complex λh Plane 127
7.6.1 Stability for Large h 127
7.6.2 Unconditional Stability, A-Stable Methods 128
7.6.3 Stability Contours in the Complex λh Plane 130
7.7 Fourier Stability Analysis 133
7.7.1 The Basic Procedure 133
7.7.2 Some Examples 134
7.7.3 Relation to Circulant Matrices 135
7.8 Consistency 135
Exercises 138

8. Choosing a Time-Marching Method 141
8.1 Stiffness Definition for ODE's 141
8.1.1 Relation to λ-Eigenvalues 141
8.1.2 Driving and Parasitic Eigenvalues 142
8.1.3 Stiffness Classifications 143
8.2 Relation of Stiffness to Space Mesh Size 143
8.3 Practical Considerations for Comparing Methods 144
8.4 Comparing the Efficiency of Explicit Methods 145
8.4.1 Imposed Constraints 145
8.4.2 An Example Involving Diffusion 146
8.4.3 An Example Involving Periodic Convection 147
8.5 Coping with Stiffness 149
8.5.1 Explicit Methods 149
8.5.2 Implicit Methods 150
8.5.3 A Perspective 151
8.6 Steady Problems 151
Exercises 152

9. Relaxation Methods 153
9.1 Formulation of the Model Problem 154
9.1.1 Preconditioning the Basic Matrix 154
9.1.2 The Model Equations 156
9.2 Classical Relaxation 157
9.2.1 The Delta Form of an Iterative Scheme 157
9.2.2 The Converged Solution, the Residual, and the Error 158
9.2.3 The Classical Methods 158
9.3 The ODE Approach to Classical Relaxation 159
9.3.1 The Ordinary Differential Equation Formulation 159
9.3.2 ODE Form of the Classical Methods 161
9.4 Eigensystems of the Classical Methods 162
9.4.1 The Point-Jacobi System 163

9.4.2 The Gauss–Seidel System 166
9.4.3 The SOR System 169
9.5 Nonstationary Processes 171
Exercises 176

10. Multigrid 177
10.1 Motivation 177
10.1.1 Eigenvector and Eigenvalue Identification with Space Frequencies 177
10.1.2 Properties of the Iterative Method 178
10.2 The Basic Process 178
10.3 A Two-Grid Process 185
Exercises 187

11. Numerical Dissipation 189
11.1 One-Sided First-Derivative Space Differencing 189
11.2 The Modified Partial Differential Equation 190
11.3 The Lax–Wendroff Method 192
11.4 Upwind Schemes 195
11.4.1 Flux-Vector Splitting 196
11.4.2 Flux-Difference Splitting 198
11.5 Artificial Dissipation 199
Exercises 200

12. Split and Factored Forms 203
12.1 The Concept 203
12.2 Factoring Physical Representations – Time Splitting 204
12.3 Factoring Space Matrix Operators in 2D 206
12.3.1 Mesh Indexing Convention 206
12.3.2 Data-Bases and Space Vectors 206
12.3.3 Data-Base Permutations 207
12.3.4 Space Splitting and Factoring 207
12.4 Second-Order Factored Implicit Methods 211
12.5 Importance of Factored Forms in Two and Three Dimensions 212
12.6 The Delta Form 213
Exercises 214

13. Analysis of Split and Factored Forms 217
13.1 The Representative Equation for Circulant Operators 217
13.2 Example Analysis of Circulant Systems 218
13.2.1 Stability Comparisons of Time-Split Methods 218
13.2.2 Analysis of a Second-Order Time-Split Method 220
13.3 The Representative Equation for Space-Split Operators 222
13.4 Example Analysis of the 2D Model Equation 225

13.4.1 The Unfactored Implicit Euler Method 225
13.4.2 The Factored Nondelta Form of the Implicit Euler Method 226
13.4.3 The Factored Delta Form of the Implicit Euler Method 227
13.4.4 The Factored Delta Form of the Trapezoidal Method . 227
13.5 Example Analysis of the 3D Model Equation 228
Exercises 230

Appendices 231

A. Useful Relations from Linear Algebra 231
A.1 Notation 231
A.2 Definitions 232
A.3 Algebra 232
A.4 Eigensystems 233
A.5 Vector and Matrix Norms 235

B. Some Properties of Tridiagonal Matrices 237
B.1 Standard Eigensystem for Simple Tridiagonal Matrices 237
B.2 Generalized Eigensystem for Simple Tridiagonal Matrices 238
B.3 The Inverse of a Simple Tridiagonal Matrix 239
B.4 Eigensystems of Circulant Matrices 240
B.4.1 Standard Tridiagonal Matrices 240
B.4.2 General Circulant Systems 241
B.5 Special Cases Found from Symmetries 241
B.6 Special Cases Involving Boundary Conditions 242

C. The Homogeneous Property of the Euler Equations 245

Index 247