

**Mathematical
Surveys
and
Monographs**
Volume 163

The Ricci Flow: Techniques and Applications

Part III: Geometric-Analytic
Aspects

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