

Contents

Preface xi

1	Elementary Topics	1
1.1	Arithmetic	2
1.2	Computation	4
1.3	Algebra	7
1.4	Geometry	9
1.5	Calculus	13
1.6	Combinatorics	16
1.7	Probability	20
1.8	Logic	22
1.9	Historical Remarks	25
1.10	Philosophical Remarks	32
2	Arithmetic	35
2.1	The Euclidean Algorithm	36
2.2	Continued Fractions	38
2.3	Prime Numbers	40
2.4	Finite Arithmetic	44
2.5	Quadratic Integers	46
2.6	The Gaussian Integers	49
2.7	Euler's Proof Revisited	54
2.8	$\sqrt{2}$ and the Pell Equation	57
2.9	Historical Remarks	60
2.10	Philosophical Remarks	67
3	Computation	73
3.1	Numerals	74
3.2	Addition	77
3.3	Multiplication	79
3.4	Division	82
3.5	Exponentiation	84

3.6	P and NP Problems	87
3.7	Turing Machines	90
3.8	*Unsolvable Problems	94
3.9	*Universal Machines	97
3.10	Historical Remarks	98
3.11	Philosophical Remarks	103
4	Algebra	106
4.1	Classical Algebra	107
4.2	Rings	112
4.3	Fields	117
4.4	Two Theorems Involving Inverses	120
4.5	Vector Spaces	123
4.6	Linear Dependence, Basis, and Dimension	126
4.7	Rings of Polynomials	128
4.8	Algebraic Number Fields	133
4.9	Number Fields as Vector Spaces	136
4.10	Historical Remarks	139
4.11	Philosophical Remarks	143
5	Geometry	148
5.1	Numbers and Geometry	149
5.2	Euclid's Theory of Angles	150
5.3	Euclid's Theory of Area	153
5.4	Straightedge and Compass Constructions	159
5.5	Geometric Realization of Algebraic Operations	161
5.6	Algebraic Realization of Geometric Constructions	164
5.7	Vector Space Geometry	168
5.8	Introducing Length via the Inner Product	171
5.9	Constructible Number Fields	175
5.10	Historical Remarks	177
5.11	Philosophical Remarks	184
6	Calculus	193
6.1	Geometric Series	194
6.2	Tangents and Differentiation	197

6.3	Calculating Derivatives	202
6.4	Curved Areas	208
6.5	The Area under $y = x^n$	211
6.6	*The Fundamental Theorem of Calculus	214
6.7	Power Series for the Logarithm	218
6.8	*The Inverse Tangent Function and π	226
6.9	Elementary Functions	229
6.10	Historical Remarks	233
6.11	Philosophical Remarks	239
7	Combinatorics	243
7.1	The Infinitude of Primes	244
7.2	Binomial Coefficients and Fermat's Little Theorem	245
7.3	Generating Functions	246
7.4	Graph Theory	250
7.5	Trees	252
7.6	Planar Graphs	254
7.7	The Euler Polyhedron Formula	256
7.8	Nonplanar Graphs	263
7.9	*The König Infinity Lemma	264
7.10	Sperner's Lemma	268
7.11	Historical Remarks	272
7.12	Philosophical Remarks	274
8	Probability	279
8.1	Probability and Combinatorics	280
8.2	Gambler's Ruin	282
8.3	Random Walk	284
8.4	Mean, Variance, and Standard Deviation	286
8.5	*The Bell Curve	290
8.6	Historical Remarks	292
8.7	Philosophical Remarks	296
9	Logic	298
9.1	Propositional Logic	299
9.2	Tautologies, Identities, and Satisfiability	302

9.3	Properties, Relations, and Quantifiers	304
9.4	Induction	307
9.5	*Peano Arithmetic	311
9.6	*The Real Numbers	315
9.7	*Infinity	320
9.8	*Set Theory	324
9.9	*Reverse Mathematics	327
9.10	Historical Remarks	330
9.11	Philosophical Remarks	333
10	Some Advanced Mathematics	336
10.1	Arithmetic: the Pell Equation	337
10.2	Computation: the Word Problem	344
10.3	Algebra: the Fundamental Theorem	349
10.4	Geometry: the Projective Line	354
10.5	Calculus: Wallis's Product for π	360
10.6	Combinatorics: Ramsey Theory	365
10.7	Probability: de Moivre's Distribution	369
10.8	Logic: the Completeness Theorem	376
10.9	Historical and Philosophical Remarks	381

Bibliography 395

Index 405