Contents

Foreword xiii

Preface xv

Introduction xvii

Index of notation xix

Part 1. A Quick Introduction to Complex Analytic Functions

Chapter 1. The complex exponential function 3

§1.1. The series 3

§1.2. The function exp is **C**-derivable 4

§1.3. The exponential function as a covering map 7

§1.4. The exponential of a matrix 8

§1.5. Application to differential equations 10

Exercises 12

Chapter 2. Power series 15

§2.1. Formal power series 15

§2.2. Convergent power series 20

§2.3. The ring of power series 22

§2.4. **C**-derivability of power series 23

§2.5. Expansion of a power series at a point $\neq 0$ 25

§2.6. Power series with values in a linear space 26

Exercises 27

Chapter 3. Analytic functions 29
§3.1. Analytic and holomorphic functions 29
§3.2. Singularities 32
§3.3. Cauchy theory 33
§3.4. Our first differential algebras 36
Exercises 37

Chapter 4. The complex logarithm 39
§4.1. Can one invert the complex exponential function? 39
§4.2. The complex logarithm via trigonometry 40
§4.3. The complex logarithm as an analytic function 41
§4.4. The logarithm of an invertible matrix 42
Exercises 44

Chapter 5. From the local to the global 45
§5.1. Analytic continuation 45
§5.2. Monodromy 47
§5.3. A first look at differential equations with a singularity 50
Exercises 52

Part 2. Complex Linear Differential Equations and their Monodromy

Chapter 6. Two basic equations and their monodromy 57
§6.1. The "characters" z^α 57
§6.2. A new look at the complex logarithm 70
§6.3. Back again to the first example 74
Exercises 75

Chapter 7. Linear complex analytic differential equations 77
§7.1. The Riemann sphere 77
§7.2. Equations of order n and systems of rank n 81
§7.3. The existence theorem of Cauchy 87
§7.4. The sheaf of solutions 89
§7.5. The monodromy representation 91
§7.6. Holomorphic and meromorphic equivalences of systems 95
Exercises 101

Chapter 8. A functorial point of view on analytic continuation: Local systems 103

§8.1. The category of differential systems on Ω 103
§8.2. The category $\mathfrak{Ls}$ of local systems on Ω 105
§8.3. A functor from differential systems to local systems 107
§8.4. From local systems to representations of the fundamental group 109
Exercises 113

Part 3. The Riemann-Hilbert Correspondence

Chapter 9. Regular singular points and the local Riemann-Hilbert correspondence 117
§9.1. Introduction and motivation 118
§9.2. The condition of moderate growth in sectors 120
§9.3. Moderate growth condition for solutions of a system 123
§9.4. Resolution of systems of the first kind and monodromy of regular singular systems 124
§9.5. Moderate growth condition for solutions of an equation 128
§9.6. Resolution and monodromy of regular singular equations 132
Exercises 135

Chapter 10. Local Riemann-Hilbert correspondence as an equivalence of categories 137
§10.1. The category of singular regular differential systems at 0 138
§10.2. About equivalences and isomorphisms of categories 139
§10.3. Equivalence with the category of representations of the local fundamental group 141
§10.4. Matricial representation 142
Exercises 144

Chapter 11. Hypergeometric series and equations 145
§11.1. Fuchsian equations and systems 145
§11.2. The hypergeometric series 149
§11.3. The hypergeometric equation 150
§11.4. Global monodromy according to Riemann 153
§11.5. Global monodromy using Barnes' connection formulas 157
Exercises 159

Chapter 12. The global Riemann-Hilbert correspondence 161
§12.1. The correspondence 161

§12.2. The twenty-first problem of Hilbert 162
Exercises 166

Part 4. Differential Galois Theory

Chapter 13. Local differential Galois theory 169
§13.1. The differential algebra generated by the solutions 170
§13.2. The differential Galois group 172
§13.3. The Galois group as a linear algebraic group 175
Exercises 179

Chapter 14. The local Schlesinger density theorem 181
§14.1. Calculation of the differential Galois group in the semi-simple case 182
§14.2. Calculation of the differential Galois group in the general case 186
§14.3. The density theorem of Schlesinger in the local setting 188
§14.4. Why is Schlesinger's theorem called a "density theorem"? 191
Exercises 192

Chapter 15. The universal (fuchsian local) Galois group 193
§15.1. Some algebra, with replicas 194
§15.2. Algebraic groups and replicas of matrices 196
§15.3. The universal group 199
Exercises 200

Chapter 16. The universal group as proalgebraic hull of the fundamental group 201
§16.1. Functoriality of the representation $\hat{\rho}_A$ of $\hat{\pi}_1$ 201
§16.2. Essential image of this functor 203
§16.3. The structure of the semi-simple component of $\hat{\pi}_1$ 207
§16.4. Rational representations of $\hat{\pi}_1$ 213
§16.5. Galois correspondence and the proalgebraic hull of $\hat{\pi}_1$ 214
Exercises 216

Chapter 17. Beyond local fuchsian differential Galois theory 219
§17.1. The global Schlesinger density theorem 220
§17.2. Irregular equations and the Stokes phenomenon 221
§17.3. The inverse problem in differential Galois theory 226

§17.4. Galois theory of nonlinear differential equations 227

Appendix A. Another proof of the surjectivity of $\exp : \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ 229

Appendix B. Another construction of the logarithm of a matrix 233

Appendix C. Jordan decomposition in a linear algebraic group 237

§C.1. Dunford-Jordan decomposition of matrices 237

§C.2. Jordan decomposition in an algebraic group 241

Appendix D. Tannaka duality without schemes 243

§D.1. One weak form of Tannaka duality 245

§D.2. The strongest form of Tannaka duality 246

§D.3. The proalgebraic hull of $\mathbf{Z}$ 248

§D.4. How to use tannakian duality in differential Galois theory 251

Appendix E. Duality for diagonalizable algebraic groups 255

§E.1. Rational functions and characters 255

§E.2. Diagonalizable groups and duality 257

Appendix F. Revision problems 259

§F.1. 2012 exam (Wuhan) 259

§F.2. 2013 exam (Toulouse) 260

§F.3. Some more revision problems 263

Bibliography 267

Index 271