

# Contents

Introduction	1
Chapter 1. Tori and Abelian Varieties	5
1. Algebraic groups	5
2. Moduli problems	15
3. Moduli of elliptic curves	18
4. Modular forms	22
5. Some examples of modular forms	26
5.1. Level 1	26
5.2. Higher level	27
6. Abelian varieties over $\mathbb{C}$	34
6.1. The Appell–Humbert Theorem	38
6.2. The dual abelian variety	42
Chapter 2. Complex Abelian Varieties with Real Multiplication and Hilbert Modular Forms	49
1. Algebraic preliminaries	50
2. Complex abelian varieties with real multiplication	51
2.1. Complex and rational representations	52
2.2. Construction of families of abelian varieties with real multiplication	53
3. Hilbert modular forms	63
4. Construction of Hilbert modular forms	68
4.1. Eisenstein series	68
4.2. Other methods of constructing modular forms	70
5. More on the diagonal curve	72
5.1. Modular interpretation	72
6. Siegel’s formula	74
Chapter 3. Abelian Varieties with Real Multiplication over General Fields	77
1. Abelian varieties over a general field	78
1.1. The dual abelian variety	79
1.2. Examples	82
1.3. Abelian schemes	83
2. Finite Heisenberg groups	84
3. Honda–Tate Theorem	90
4. Ordinary abelian varieties and Serre–Tate coordinates	93
4.1. Ordinary abelian varieties	93
4.2. Serre–Tate coordinates	94
5. Abelian varieties with real multiplication over a general field	97
6. Irreducibility of the moduli space of $\mu_{p^\infty}$ -level structure	101

6.1. Moduli spaces for full and $\mu_N$ -level	101
6.2. Deligne's description of ordinary abelian varieties	104
6.3. Irreducibility	105
<b>Chapter 4. <math>p</math>-adic Elliptic Modular Forms</b>	<b>109</b>
1. Introduction	109
1.1. $p$ -adic $L$ -functions	110
1.2. Deformation of Galois representations	113
2. Congruences between modular forms mod $p$	117
3. Operators and systems of eigenvalues	124
3.1. A highbrow view of modular forms in characteristic $p$	124
3.2. Operators	126
3.3. Filtration and systems of eigenvalues	130
3.4. Congruences mod $p^m$	132
4. Serre's $p$ -adic modular forms and $p$ -adic zeta functions	134
5. A geometric approach to congruences	137
5.1. The Hasse invariant	139
5.2. The kernel of the $q$ -expansion map	140
5.3. Operators revisited	142
6. $p$ -adic elliptic modular forms	146
6.1. Test objects and overconvergent forms	146
6.2. $q$ -expansion for $p$ -adic modular forms	149
6.3. Standing assumptions and their <i>raison d'être</i>	150
6.4. The case when $p$ is nilpotent	151
6.5. The case of $r$ a unit	153
6.6. Katz's expansion	155
6.7. Properties of $q$ -expansions of $p$ -adic modular forms	158
7. The ring of divided congruences	159
<b>Chapter 5. <math>p</math>-adic Hilbert Modular Forms</b>	<b>167</b>
1. Algebraic Hilbert modular forms	168
2. Tate objects and the $q$ -expansion	171
3. Hasse invariants	176
3.1. Definition and main properties of partial Hasse invariants	176
3.2. Further properties	180
4. The kernel of the $q$ -expansion map	181
5. Applications	184
6. $p$ -adic Hilbert modular forms	186
6.1. Test objects and overconvergent forms	186
6.2. $q$ -expansion for $p$ -adic modular forms	188
6.3. The case when $p$ is nilpotent or $r$ is a unit	189
6.4. Katz's expansion	190
6.5. Properties of $q$ -expansions of $p$ -adic modular forms	191
<b>Chapter 6. Deformation Theory of Abelian Varieties</b>	<b>193</b>
1. Fine moduli schemes	193
2. Proof of the Serre–Tate Theorem	197
3. Deformation of $p$ -divisible groups	199
4. Commutative smooth formal groups	201

4.1. Curves	202
4.2. Formal groups	203
4.3. Operators on $\mathcal{C}(\mathcal{G})$	204
5. Modules over $\text{Cart}(K)$	207
6. The case of $\mathbb{Q}$ -algebras	209
6.1. Digression on Witt vectors	209
7. Formal groups in characteristic $p$	212
8. Classification of $p$ -divisible groups in characteristic $p$ , Newton polygons and types	213
9. Mid-way summary	220
10. Displays	221
10.1. Basics	222
10.2. Examples	225
10.3. The main result	226
11. The universal display and applications to abelian varieties	226
11.1. Base change and deformations	226
11.2. Universal display	226
11.3. Endomorphisms conditions	227
11.4. The local structure of $W_\tau$ and Hasse invariants	229
11.5. Polarization conditions	231
Appendix A. Group Schemes	235
1. Some definitions	235
2. Digression on Frobenius and Verschiebung	237
3. Important examples	238
3.1. The multiplicative group $\mathbb{G}_m$	238
3.2. The roots of unity $\mu_N$	238
3.3. The additive group $\mathbb{G}_a$	239
3.4. The group $\alpha_{p^r}$	239
3.5. The group $\text{GL}_n$	239
3.6. A non-commutative group scheme of rank $p^2$	240
3.7. The constant group scheme $\underline{\Gamma}$	240
3.8. Étale group schemes	242
3.9. The $p$ -torsion group scheme $A[p]$	242
4. The basic exact sequence	244
5. Group schemes over a perfect field of characteristic $p$	245
6. The $\alpha$ -group	248
Appendix B. Calculating with Cusps	251
1. $\Gamma_1(N)$ -level	252
2. $\Gamma_0(N)$ -level	253
3. $\Gamma(N)$ -level	254
Bibliography	255
Notation Index	263
Index	267