

# Contents

Preface	v
Using This Book	ix
<b>Part I: Finite Groups</b>	<b>1</b>
1. Representations of Finite Groups	3
§1.1: Definitions	3
§1.2: Complete Reducibility; Schur's Lemma	5
§1.3: Examples: Abelian Groups; $\mathfrak{S}_3$	8
2. Characters	12
§2.1: Characters	12
§2.2: The First Projection Formula and Its Consequences	15
§2.3: Examples: $\mathfrak{S}_4$ and $\mathfrak{A}_4$	18
§2.4: More Projection Formulas; More Consequences	21
3. Examples; Induced Representations; Group Algebras; Real Representations	26
§3.1: Examples: $\mathfrak{S}_5$ and $\mathfrak{A}_5$	26
§3.2: Exterior Powers of the Standard Representation of $\mathfrak{S}_4$	31
§3.3: Induced Representations	32
§3.4: The Group Algebra	36
§3.5: Real Representations and Representations over Subfields of $\mathbb{C}$	39

4. Representations of $\mathfrak{S}_d$ : Young Diagrams and Frobenius's Character Formula	44
§4.1: Statements of the Results	44
§4.2: Irreducible Representations of $\mathfrak{S}_d$	52
§4.3: Proof of Frobenius's Formula	54
5. Representations of $\mathfrak{A}_d$ and $GL_2(\mathbb{F}_q)$	63
§5.1: Representations of $\mathfrak{A}_d$	63
§5.2: Representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$	67
6. Weyl's Construction	75
§6.1: Schur Functors and Their Characters	75
§6.2: The Proofs	84
<b>Part II: Lie Groups and Lie Algebras</b>	<b>89</b>
7. Lie Groups	93
§7.1: Lie Groups: Definitions	93
§7.2: Examples of Lie Groups	95
§7.3: Two Constructions	101
8. Lie Algebras and Lie Groups	104
§8.1: Lie Algebras: Motivation and Definition	104
§8.2: Examples of Lie Algebras	111
§8.3: The Exponential Map	114
9. Initial Classification of Lie Algebras	121
§9.1: Rough Classification of Lie Algebras	121
§9.2: Engel's Theorem and Lie's Theorem	125
§9.3: Semisimple Lie Algebras	128
§9.4: Simple Lie Algebras	131
10. Lie Algebras in Dimensions One, Two, and Three	133
§10.1: Dimensions One and Two	133
§10.2: Dimension Three, Rank 1	136
§10.3: Dimension Three, Rank 2	139
§10.4: Dimension Three, Rank 3	141
11. Representations of $\mathfrak{sl}_2\mathbb{C}$	146
§11.1: The Irreducible Representations	146
§11.2: A Little Plethysm	151
§11.3: A Little Geometric Plethysm	153

Contents	xiii
12. Representations of $\mathfrak{sl}_3\mathbb{C}$ , Part I	161
13. Representations of $\mathfrak{sl}_3\mathbb{C}$ , Part II: Mainly Lots of Examples	175
§13.1: Examples	175
§13.2: Description of the Irreducible Representations	182
§13.3: A Little More Plethysm	185
§13.4: A Little More Geometric Plethysm	189
<b>Part III: The Classical Lie Algebras and Their Representations</b>	<b>195</b>
14. The General Set-up: Analyzing the Structure and Representations of an Arbitrary Semisimple Lie Algebra	197
§14.1: Analyzing Simple Lie Algebras in General	197
§14.2: About the Killing Form	206
15. $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$	211
§15.1: Analyzing $\mathfrak{sl}_n\mathbb{C}$	211
§15.2: Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$	217
§15.3: Weyl's Construction and Tensor Products	222
§15.4: Some More Geometry	227
§15.5: Representations of $GL_n\mathbb{C}$	231
16. Symplectic Lie Algebras	238
§16.1: The Structure of $Sp_{2n}\mathbb{C}$ and $sp_{2n}\mathbb{C}$	238
§16.2: Representations of $sp_4\mathbb{C}$	244
17. $sp_6\mathbb{C}$ and $sp_{2n}\mathbb{C}$	253
§17.1: Representations of $sp_6\mathbb{C}$	253
§17.2: Representations of $sp_{2n}\mathbb{C}$ in General	259
§17.3: Weyl's Construction for Symplectic Groups	262
18. Orthogonal Lie Algebras	267
§18.1: $SO_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$	267
§18.2: Representations of $\mathfrak{so}_3\mathbb{C}$ , $\mathfrak{so}_4\mathbb{C}$ , and $\mathfrak{so}_5\mathbb{C}$	273
19. $\mathfrak{so}_6\mathbb{C}$ , $\mathfrak{so}_7\mathbb{C}$ , and $\mathfrak{so}_m\mathbb{C}$	282
§19.1: Representations of $\mathfrak{so}_6\mathbb{C}$	282
§19.2: Representations of the Even Orthogonal Algebras	286
§19.3: Representations of $\mathfrak{so}_7\mathbb{C}$	292
§19.4: Representations of the Odd Orthogonal Algebras	294
§19.5: Weyl's Construction for Orthogonal Groups	296

20. Spin Representations of $so_m \mathbb{C}$	299
§20.1: Clifford Algebras and Spin Representations of $so_m \mathbb{C}$	299
§20.2: The Spin Groups $Spin_m \mathbb{C}$ and $Spin_m \mathbb{R}$	307
§20.3: $Spin_8 \mathbb{C}$ and Triality	312
<b>Part IV: Lie Theory</b>	
21. The Classification of Complex Simple Lie Algebras	317
§21.1: Dynkin Diagrams Associated to Semisimple Lie Algebras	319
§21.2: Classifying Dynkin Diagrams	325
§21.3: Recovering a Lie Algebra from Its Dynkin Diagram	330
22. $\mathfrak{g}_2$ and Other Exceptional Lie Algebras	339
§22.1: Construction of $\mathfrak{g}_2$ from Its Dynkin Diagram	339
§22.2: Verifying That $\mathfrak{g}_2$ is a Lie Algebra	346
§22.3: Representations of $\mathfrak{g}_2$	350
§22.4: Algebraic Constructions of the Exceptional Lie Algebras	359
23. Complex Lie Groups; Characters	366
§23.1: Representations of Complex Simple Groups	366
§23.2: Representation Rings and Characters	375
§23.3: Homogeneous Spaces	382
§23.4: Bruhat Decompositions	395
24. Weyl Character Formula	399
§24.1: The Weyl Character Formula	399
§24.2: Applications to Classical Lie Algebras and Groups	403
25. More Character Formulas	415
§25.1: Freudenthal's Multiplicity Formula	415
§25.2: Proof of (WCF); the Kostant Multiplicity Formula	419
§25.3: Tensor Products and Restrictions to Subgroups	424
26. Real Lie Algebras and Lie Groups	430
§26.1: Classification of Real Simple Lie Algebras and Groups	430
§26.2: Second Proof of Weyl's Character Formula	440
§26.3: Real, Complex, and Quaternionic Representations	444
<b>Appendices</b>	
A. On Symmetric Functions	451
§A.1: Basic Symmetric Polynomials and Relations among Them	453
§A.2: Proofs of the Determinantal Identities	462
§A.3: Other Determinantal Identities	465

<b>B. On Multilinear Algebra</b>	<b>471</b>
§B.1: Tensor Products	471
§B.2: Exterior and Symmetric Powers	472
§B.3: Duals and Contractions	475
<b>C. On Semisimplicity</b>	<b>478</b>
§C.1: The Killing Form and Cartan's Criterion	478
§C.2: Complete Reducibility and the Jordan Decomposition	481
§C.3: On Derivations	483
<b>D. Cartan Subalgebras</b>	<b>487</b>
§D.1: The Existence of Cartan Subalgebras	487
§D.2: On the Structure of Semisimple Lie Algebras	489
§D.3: The Conjugacy of Cartan Subalgebras	491
§D.4: On the Weyl Group	493
<b>E. Ado's and Levi's Theorems</b>	<b>499</b>
§E.1: Levi's Theorem	499
§E.2: Ado's Theorem	500
<b>F. Invariant Theory for the Classical Groups</b>	<b>504</b>
§F.1: The Polynomial Invariants	504
§F.2: Applications to Symplectic and Orthogonal Groups	511
§F.3: Proof of Capelli's Identity	514
<b>Hints, Answers, and References</b>	<b>516</b>
<b>Bibliography</b>	<b>536</b>
<b>Index of Symbols</b>	<b>543</b>
<b>Index</b>	<b>547</b>