

# Contents

<b>1</b>	<b>Preliminaries</b> .....	<b>1</b>
1.1	Probability Spaces and Random Variables .....	1
1.1.1	Probability Spaces .....	1
1.1.2	Random Variables .....	2
1.1.3	Random Vectors .....	4
1.1.4	Normal Distribution .....	5
1.1.5	Product Spaces .....	5
1.1.6	The Essential Supremum .....	6
1.2	Stochastic Processes .....	7
1.2.1	Filtrations .....	7
1.2.2	Stochastic Processes .....	8
1.2.3	Stopping Times .....	10
1.2.4	Martingales .....	11
1.2.5	Markov Processes .....	12
1.3	Some Inequalities and Convergence Theorems .....	12
1.3.1	Some Norms and Spaces .....	12
1.3.2	Some Inequalities .....	13
1.3.3	Some Convergence Theorems .....	14
1.3.4	Weak Convergence .....	14
1.3.5	Monotone Class Theorem .....	15
1.4	Exercises .....	15

## Part I The Basic Theory of SDEs and BSDEs

<b>2</b>	<b>Basics of Stochastic Calculus</b> .....	<b>21</b>
2.1	Brownian Motion .....	21
2.1.1	Definition .....	21
2.1.2	Pathwise Properties .....	22
2.1.3	The Augmented Filtration .....	25

- 2.2 Stochastic Integration ..... 26
  - 2.2.1 Some Heuristic Arguments..... 26
  - 2.2.2 Itô Integral for Elementary Processes ..... 27
  - 2.2.3 Itô Integral in  $L^2(\mathbb{F})$  and  $L^2_{loc}(\mathbb{F})$  ..... 30
- 2.3 The Itô Formula ..... 32
  - 2.3.1 Some Heuristic Arguments..... 32
  - 2.3.2 The Itô Formula..... 33
  - 2.3.3 Itô Formula in Multidimensional Case ..... 38
  - 2.3.4 An Extended Itô Formula ..... 39
- 2.4 The Burkholder-Davis-Gundy Inequality ..... 39
- 2.5 The Martingale Representation Theorem ..... 42
- 2.6 The Girsanov Theorem..... 46
- 2.7 The Doob-Meyer Decomposition ..... 50
- 2.8 A Financial Application..... 52
  - 2.8.1 Pricing via Risk Neutral Measure..... 52
  - 2.8.2 Hedging the Option..... 53
  - 2.8.3 Some Further Discussion..... 55
- 2.9 Bibliographical Notes ..... 56
- 2.10 Exercises ..... 57
- 3 Stochastic Differential Equations ..... 63**
  - 3.1 Linear Stochastic Differential Equations ..... 63
  - 3.2 A Priori Estimates for SDEs ..... 65
  - 3.3 Well-Posedness of SDEs ..... 68
  - 3.4 Some Properties of SDEs ..... 70
  - 3.5 Weak Solutions of SDEs ..... 73
  - 3.6 Bibliographical Notes ..... 76
  - 3.7 Exercises ..... 76
- 4 Backward Stochastic Differential Equations ..... 79**
  - 4.1 Linear Backward Stochastic Differential Equations ..... 80
  - 4.2 A Priori Estimates for BSDEs ..... 82
  - 4.3 Well-Posedness of BSDEs ..... 84
  - 4.4 Basic Properties of BSDEs ..... 87
  - 4.5 Some Applications of BSDEs ..... 90
    - 4.5.1 Application in Asset Pricing and Hedging Theory..... 91
    - 4.5.2 Applications in Stochastic Control ..... 92
  - 4.6 Bibliographical Notes ..... 95
  - 4.7 Exercises ..... 96
- 5 Markov BSDEs and PDEs ..... 101**
  - 5.1 Markov Property and Nonlinear Feynman-Kac Formula..... 102
    - 5.1.1 Markov SDEs ..... 102
    - 5.1.2 Markov BSDEs ..... 104
    - 5.1.3 Nonlinear Feynman-Kac Formula ..... 105
  - 5.2 Regularity of Solutions..... 106

5.3	Time Discretization of SDEs and BSDEs .....	112
5.3.1	Euler Scheme for SDEs .....	112
5.3.2	Backward Euler Scheme for BSDEs.....	113
5.4	Implementation of Backward Euler Scheme .....	117
5.4.1	Least Square Regression .....	118
5.4.2	Monte Carlo Simulation .....	119
5.5	Viscosity Property of BSDEs .....	120
5.6	Bibliographical Notes .....	127
5.7	Exercises .....	128

## Part II Further Theory of BSDEs

<b>6</b>	<b>Reflected Backward SDEs</b> .....	133
6.1	American Options and Reflected BSDEs .....	133
6.2	A Priori Estimates .....	137
6.3	Well-Posedness of RBSDEs .....	141
6.3.1	The Snell Envelope Theory .....	141
6.3.2	Existence via Picard Iteration .....	147
6.3.3	Existence via Penalization .....	149
6.4	Markov RBSDEs and Obstacle Problem of PDEs .....	152
6.5	Semilinear Doob-Meyer Decomposition .....	158
6.6	Bibliographical Notes .....	158
6.7	Exercises .....	159
<b>7</b>	<b>BSDEs with Quadratic Growth in <math>Z</math></b> .....	161
7.1	Introduction.....	161
7.2	BMO Martingales and A Priori Estimates .....	163
7.3	Well-Posedness .....	167
7.4	Bibliographical Notes .....	174
7.5	Exercises.....	175
<b>8</b>	<b>Forward-Backward SDEs</b> .....	177
8.1	Introduction.....	177
8.2	Well-Posedness in Small Time Duration .....	180
8.3	The Decoupling Approach .....	184
8.3.1	The Four Step Scheme.....	184
8.3.2	The Decoupling Field .....	186
8.3.3	A Sufficient Condition for the Existence of Decoupling Field .....	188
8.4	The Method of Continuation.....	194
8.5	Bibliographical Notes .....	199
8.6	Exercises.....	200

## Part III The Fully Nonlinear Theory of BSDEs

<b>9</b>	<b>Stochastic Calculus Under Weak Formulation</b> .....	205
9.1	Some Motivations for Weak Formulation .....	205
9.1.1	Practical Considerations on Information .....	206

9.1.2	Stochastic Controls .....	208
9.1.3	Two Person Zero-Sum Stochastic Differential Games....	209
9.2	The Canonical Setting and Semimartingale Measures .....	213
9.2.1	The Canonical Setting .....	213
9.2.2	Semimartingale Measures .....	214
9.2.3	Weak Compactness .....	220
9.2.4	The Localized Spaces .....	225
9.3	Regular Conditional Probability Distributions .....	225
9.3.1	The Shifting Operators .....	225
9.3.2	Regular Conditional Probability Distribution .....	226
9.3.3	Dynamic Sets of Probability Measures .....	230
9.4	Functional Itô Formula .....	234
9.5	Bibliographical Notes .....	240
9.6	Exercises .....	241
<b>10</b>	<b>Nonlinear Expectation .....</b>	<b>245</b>
10.1	Nonlinear Expectation .....	245
10.1.1	Convergence Under Nonlinear Expectation .....	247
10.1.2	Quasi-Sure Continuity .....	250
10.1.3	Some Hitting Times .....	252
10.2	Pathwise Conditional Nonlinear Expectation .....	258
10.3	Optimal Stopping Under Nonlinear Expectation .....	262
10.3.1	Regularity and Dynamic Programming Principle .....	263
10.3.2	Local Pathwise $\mathcal{E}^{\mathcal{P}}$ -Martingale Property .....	267
10.3.3	Continuous Approximation of Stopping Times .....	268
10.3.4	Proof of Theorem 10.3.2 .....	271
10.4	Bibliographical Notes .....	273
10.5	Exercises .....	274
<b>11</b>	<b>Path Dependent PDEs .....</b>	<b>277</b>
11.1	The Viscosity Theory of Path Dependent Heat Equations .....	278
11.1.1	Classical Solutions .....	278
11.1.2	Definition of Viscosity Solutions .....	280
11.1.3	Well-Posedness in the Sense of Viscosity Solutions .....	281
11.2	Viscosity Solution of General Parabolic PPDEs .....	283
11.2.1	Definition of Viscosity Solutions .....	284
11.2.2	Consistency with Classical Solutions .....	286
11.2.3	Equivalent Definition via Semijets .....	288
11.2.4	A Change Variable Formula .....	290
11.3	Examples of PPDEs .....	290
11.3.1	First Order PPDEs .....	290
11.3.2	Semilinear PPDEs .....	292
11.3.3	Path Dependent HJB Equations .....	295
11.3.4	Path Dependent Isaacs Equations .....	300
11.3.5	Stochastic HJB Equations and Backward Stochastic PDEs .....	307

11.4	Well-Posedness of Fully Nonlinear PPDEs .....	310
11.4.1	Stability .....	311
11.4.2	Partial Comparison of Viscosity Solutions .....	313
11.4.3	Comparison Principle of PPDEs .....	314
11.5	Monotone Scheme for PPDEs .....	320
11.5.1	Monotone Scheme for PDEs .....	320
11.5.2	Monotone Scheme for PPDEs .....	323
11.5.3	Discretization of the Nonlinear Expectation .....	329
11.6	Bibliographical Notes .....	332
11.7	Exercises .....	333
<b>12</b>	<b>Second Order BSDEs .....</b>	<b>335</b>
12.1	Quasi-Sure Stochastic Analysis .....	335
12.1.1	Quasi-Sure Stochastic Integration .....	336
12.1.2	Quasi-Sure Conditional Nonlinear Expectation .....	338
12.2	Second Order BSDEs .....	345
12.2.1	Representation and Uniqueness .....	349
12.2.2	A Priori Estimates .....	351
12.2.3	Existence .....	355
12.3	Extension to the Case with Measurable Coefficients .....	359
12.4	An Application in an Uncertain Volatility Model .....	361
12.5	Bibliographical Notes .....	363
12.6	Exercises .....	364
	<b>References .....</b>	<b>365</b>
	<b>Frequently Used Notation .....</b>	<b>377</b>
	<b>Index .....</b>	<b>383</b>