

TABLE OF CONTENTS

PREFACE	xxv
BASIC SYMBOLS	1
BASIC NOTATIONS	2
 CHAPTER I	
EULER'S φ -FUNCTION	9
§ I. 1 Elementary inequalities for φ	9
§ I. 2 Inequalities for $\varphi(mn)$	9
§ I. 3 Relations connecting φ , σ , d	10
§ I. 4 Inequalities for J_k , σ_k , ψ_k	11
§ I. 5 Unitary analogues of J_k , σ_k , d	12
§ I. 6 Composition of φ , σ , ψ	13
§ I. 7 Composition of σ , φ	13
§ I. 8 On the function $n/\varphi(n)$	14
§ I. 9 Minimum of $\varphi(n)/n$ for consecutive values of n	15
§ I. 10 On $\varphi(n+1)/\varphi(n)$	16
§ I. 11 On $(\varphi(n+1), \varphi(n))$	18
§ I. 12 On $(n, \varphi(n))$	18
§ I. 13 The difference of consecutive totients	19
§ I. 14 Nonmonotonicity of φ . (A measure)	19
§ I. 15 Nonmonotonicity of J_k	20
§ I. 16 Number of solutions of $\varphi(x) = n!$	20
§ I. 17 Number of solutions of $\varphi(x) = m$	21
§ I. 18 Number of values of φ less than or equal to x	22
§ I. 19 On composite n with $\varphi(n) (n-1)$ (Lehmer's conjecture)	23
§ I. 20 Number of composite $n \leq x$ with $\varphi(n) (n-1)$	24
§ I. 21 $\sum_{n \leq x} \varphi(n)$	24
§ I. 22 $\sum_{k \leq n} f\left(\frac{k}{n}\right) \cdot \varphi(k)$	25
§ I. 23 On $\sum_{n \leq x} \varphi(n) - \frac{3}{\pi^2} x^2$	25
§ I. 24 On $\sum_{n \leq x} \varphi(n)/n$	27

§ I.25 On $\sum_{n \leq x} J_k(n) - x^{k+1}/(k+1) \zeta(k+1)$	28
§ I.26 An expansion of J_k	29
§ I.27 On $\sum_{n \leq x} 1/\varphi(n)$ and related questions	29
§ I.28 $\sum_{p \leq x} \varphi(p-1)$ for p prime	30
§ I.29 On $\sum_{n \leq x} \varphi(f(n))$, f a polynomial	31
§ I.30 $\sum_{n \leq x} \varphi^*(n)$, $\sum_{n \leq x} \varphi(n) \varphi(n+k)$ and related results	31
§ I.31 Asymptotic formulae for generalized Euler function	32
§ I.32 On $\varphi(x, n) = \sum_{m \leq x, (m, n)=1} 1$ and on Jacobstahl's arithmetic function	33
§ I.33 On the iteration of φ	34
§ I.34 Iterates of φ and the order of $\varphi^{(k)}(n)/\varphi^{(k+1)}(n)$	35
§ I.35 Normal order of $\omega(\varphi(n))$	36

CHAPTER II

THE ARITHMETICAL FUNCTION $d(n)$, ITS GENERALIZATIONS AND ITS ANALOGUES

THE ARITHMETICAL FUNCTION $d(n)$, ITS GENERALIZATIONS AND ITS ANALOGUES	39
§ II. 1 The divisor functions at consecutive integers	39
§ II. 2 On $d(n+i_1) > \dots > d(n+i_r)$	40
§ II. 3 Relations connecting d , ω , Λ , d_k	41
§ II. 4 On $d(mn)$	42
§ II. 5 An inequality for $d_k(n)$	42
§ II. 6 Majorization for $\log d(n)/\log 2$	42
§ II. 7 $\max_{n \leq x} d(n)$ and $\max_{n \leq x} (d(n), d(n+1))$ and generalizations	44
§ II. 8 Highly composite, superior highly composite, and largely composite numbers	45
§ II. 9 Congruence property of $d(n)$	47
§ II.10 $\Delta(x) = \sum_{n \leq x} d(n) - x \log x - (2\gamma - 1)x$	47
§ II.11 $\sum_{p \leq x} d(p-1)$, p prime	49
§ II.12 $\Delta_k(x) = \sum_{n \leq x} d_k(n) - x \cdot P_{k-1}(\log x)$, $k \geq 2$	51
§ II.13 $\sum_{n \leq x} d_k^2(n)$	55
§ II.14 On $\sum_{n \leq x} (g * d_k)(n)$	55
§ II.15 $\Delta_3(x)$	56
§ II.16 The divisor problem in arithmetic progressions	57
§ II.17 On $\sum_{n \leq x} 1/d_k(n)$	59

§ II.18 Average order of $d_k(n)$ over integers free of large prime factors	60
§ II.19 On a sum on d_k and Legendre's symbol	60
§ II.20 A sum on d_k , d and ω	61
§ II.21 On $\sum_{n \leq x} d(n) \cdot d(n+N)$ and related problems	61
§ II.22 On $\sum_{n \leq x} d_k(n) \cdot d(n+1)$ and related questions	63
§ II.23 Iteration of d	65
§ II.24 On $d(f(n))$ and $d(d(f(n)))$, f a polynomial	66
§ II.25 On $\sum_{n \leq x} d(n^2 + a)$ and $\sum_{m, n \leq x} d(m^2 + n^2)$	67
§ II.26 $\sum_{ f(r, s) \leq N} d(f(r, s))$, $f(x, y)$ a binary cubic form	68
§ II.27 Weighted divisor problem	68
§ II.28 On $\sum_{k < n^{1/4}} d(n - k^2)$	69
§ II.29 Divisor sums on squarefree or squarefull integers	69
§ II.30 Exponential divisors	71
§ II.31 Bi-unitary divisors	72
§ II.32 Sums over $d(n) \cdot \omega(n)$, $d(n)/\omega(n)$, $\omega(d(n))$, $\Omega(d(n))$	72
§ II.33 $\sum_{n \leq x} d(a(n))$, $a(n)$ the number of abelian groups with n elements	73
§ II.34 $d(n)$ in short intervals	73
§ II.35 Number of distinct values of $d(n)$ for $1 \leq n \leq x$	74
§ II.36 On the distribution function of $d(n)$	74
§ II.37 On $(nd(n), \sigma(n)) = 1$	75
§ II.38 Average value for the number of divisors of sums $a + b$	75
CHAPTER III	
SUM-OF-DIVISORS FUNCTION, GENERALIZATIONS, ANALOGUES; PERFECT NUMBERS AND RELATED PROBLEMS	77
§ III. 1 Elementary inequalities on $\sigma(n)$ and $\sigma(n)/n$	77
§ III. 2 On $\sigma(n)/n \log \log n$	79
§ III. 3 On $\sigma_k(n)/n^k$	80
§ III. 4 $\sum_{n \leq x} \sigma(n)$, $\sum_{n \leq x, p n} \sigma(n)$, $\sum_{n \leq x, (n, k)=1} \sigma(n)$	81
§ III. 5 Sums over $\frac{\sigma(n)}{n}$	82
§ III. 6 Sums over $\sigma^k(n)$	83
§ III. 7 On sums over $\sigma_{-\alpha}(f(n))$, f a polynomial ($0 < \alpha < 1$)	84

§ III. 8 On $\sum_{n \leq x} \sigma(f(n))$, f a polynomial	85
§ III. 9 Sums on $\sigma_\alpha(n)$, $\sigma_\beta(n+k)$	85
§ III.10 Inequalities connecting σ_k , d , γ , Ω	86
§ III.11 Sums over $\sigma(p-1)$, p a prime	87
§ III.12 On $\sigma(mn)$	87
§ III.13 On $\sigma(n) \geq 4\varphi(n)$	88
§ III.14 On $\sigma(n+i)/\sigma(n+i-1)$ and related theorems	88
§ III.15 On $\sigma(\sigma(n))$; $\sigma^*(\sigma^*(n))$ and $\sigma^{(k)}(n)$, $\sigma(\varphi(n))$, $\varphi(\sigma(n))$	89
§ III.16 Divisibility properties of $\sigma_k(n)$	91
§ III.17 Divisibility and congruences properties of $\sigma_k(n)$	92
§ III.18 On $s(n) = \sigma(n) - n$	93
§ III.19 Number of distinct values of $\sigma(n)/n$, $n \leq x$	94
§ III.20 Frequency of integers $m \leq N$ with $\log(\varphi(m)/m) \leq x$, $\log(\sigma(m)/m) \leq y$	95
§ III.21 On $\frac{\sigma(a^n-1)}{a^n-1}$ and related functions	95
§ III.22 Normal order of $\omega(\sigma_k(n))$	96
§ III.23 Number of prime factors of $(\sigma(A_k), A_k)$	97
§ III.24 On $\sigma(p^a) = x^b$	97
§ III.25 An inequality for $\sigma^*(n)$	97
§ III.26 Sums over $\sigma^*(n)$, $\frac{1}{\log \sigma^*(n)}$, $\sigma_k^{*2}(n)$	98
§ III.27 Inequalities on σ_k^* , d^* , σ , ψ	99
§ III.28 The sum of exponential divisors	99
§ III.29 Average order of $\sigma^\epsilon(n)$	100
§ III.30 Number of distinct prime divisors of an odd perfect number	100
§ III.31 Bounds for the prime divisors of an odd perfect number	102
§ III.32 Density of perfect numbers	104
§ III.33 Multiply perfect and multiperfect numbers	105
§ III.34 k -perfect numbers	106
§ III.35 Primitive abundant numbers	107
§ III.36 Deficient numbers	108
§ III.37 Triperfect numbers	108
§ III.38 Quasiperfect numbers	109
§ III.39 Almost perfect numbers	110
§ III.40 Superperfect numbers	110
§ III.41 Superabundant and highly abundant numbers	111
§ III.42 Amicable numbers	112
§ III.43 Weird numbers	113

§ III.44 Hyperperfect numbers	114
§ III.45 Unitary perfect numbers, bi-unitary perfect numbers	114
§ III.46 Primitive unitary abundant numbers	115
§ III.47 Nonunitary perfect numbers	116
§ III.48 Exponentially perfect numbers	116
§ III.49 Exponentially, powerful perfect numbers	117
§ III.50 Practical numbers	118
§ III.51 Unitary harmonic numbers	119
§ III.52 Perfect Gaussian integers	120
 CHAPTER IV	
P, p, B, β AND RELATED FUNCTIONS	121
§ IV. 1 Sums over $P(n)$, $p(n)$, $P(n)/p(n)$, $1/P'(n)$	121
§ IV. 2 Sums over $\log P(n)$	122
§ IV. 3 Sums over $P(n)^{-\omega(n)}$ and $P(n)^{-\Omega(n)}$	123
§ IV. 4 Sums on $1/p(n)$, $\omega(n)/p(n)$, $d(n)/p(n)$	123
§ IV. 5 Density of reducible integers	124
§ IV. 6 On $p(n! + 1)$, $P(n! + 1)$, $P(F_n)$	125
§ IV. 7 Greatest prime factor of an arithmetic progression	125
§ IV. 8 $P(n^2 + 1)$ and $P(n^4 + 1)$	126
§ IV. 9 $P(a^n - b^n)$, $P(a^p - b^p)$	127
§ IV.10 $P(u_n)$ for a recurrence sequence (u_n)	128
§ IV.11 Greatest prime factor of a product	129
§ IV.12 $P(f(x))$, f a polynomial	130
§ IV.13 Greatest prime factor of a quadratic polynomial	131
§ IV.14 $P(p+a)$, $p(p+a)$, p prime	132
§ IV.15 On $P(ax^m + by^n)$	132
§ IV.16 Intervals containing numbers without large prime factors	133
§ IV.17 On $P(n)/P(n+1)$	134
§ IV.18 Consecutive prime divisors	135
§ IV.19 Greatest prime factor of consecutive integers	135
§ IV.20 Frequency of numbers containing prime factors of a certain relative magnitude	136
§ IV.21 Integers without large prime factors. The function $\psi(x,y)$ and Dickman's function	136
§ IV.22 Function $\psi(x,y;a,q)$. Integers without large prime factors in arithmetic progressions.	141
§ IV.23 On $(n, \beta(n)) = 1$	143
§ IV.24 Sums over $\beta_k(n)$, $B_k(n)$, $B(n) - \beta(n)$, $\frac{B(n)}{\beta(n)}$, $\frac{B(n) - \beta(n)}{P(n)}$	143

§ IV.25 Sums over $\frac{\beta(n)}{P(n)}, \frac{P(n)}{\beta(n)}, B(n) - P_1(n) - \dots - P_{n-1}(n)$	145
§ IV.26 Distribution of $\frac{B(n)}{\beta(n)}$	146
§ IV.27 On $(-1)^{B(n)}$	146
§ IV.28 Sums over $B^1(n), P(n)/B^1(n), B^1(n)/B(n), 1/B^1(n)$, etc.	147
§ IV.29 Numbers n with the property $B(n) = B(n+1)$	148
§ IV.30 On greatest prime divisors of sums of integers	149
§ IV.31 On $\sum_{n \leq x} f(P(n)), f$ a certain arithmetic function	150
§ IV.32 On $\Phi(x,y)$ and Buchstab's function	151
§ IV.33 On the partition of primes into two subsets with nearly the same number of products	153

CHAPTER V

$\omega(n), \Omega(n)$ AND RELATED FUNCTIONS	155
§ V. 1 Average order of $\omega, \Omega, \Omega - \omega, \Omega_k$	155
§ V. 2 Sums over $\omega^2(n), \omega^k(n)$	155
§ V. 3 Sums over $(\omega(n) - \log \log x)^2$	156
§ V. 4 $\sum_{2 \leq n \leq x} \frac{1}{\omega(n)}, \sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)}$, etc.	157
§ V. 5 $\sum_{p \leq n} \omega^k(p-1)$ (p prime)	159
§ V. 6 $\sum_{p \leq n} \omega(f(p), f$ polynomial (p prime)	160
§ V. 7 $\sum_{n \leq x} z^{\omega(n)}$ and related sums	161
§ V. 8 Sums over $\lambda(n) = (-1)^{\Omega(n)}$	162
§ V. 9 Sums over $n^{-1/\omega(n)}, n^{-1/\Omega(n)}$	162
§ V.10 Sums on $d(n)\omega(n-1), d_k(n)\omega(n)$	163
§ V.11 Sums on $\frac{\omega(n)}{P(n)}, \frac{\omega(n)}{\beta(n)}$	163
§ V.12 $\omega(a(n)), \omega(d(n)),$ etc.	164
§ V.13 $\frac{\Omega(n) - \omega(n)}{P(n)}, \frac{\Omega(n) - \omega(n)}{\beta(n)}$, etc.	165
§ V.14 On the number of integers $n \leq x$ with $\Omega(n) - \omega(n) = k$	165
§ V.15 Estimates of type $\omega(n) \leq c \cdot \log n / \log \log n$	167
§ V.16 On $\omega(n) - \omega(n+1)$ or $\omega(m) - \omega(n)$	168
§ V.17 The values of ω on consecutive integers	169
§ V.18 Local growth of ω at consecutive integers	170
§ V.19 Normal order of $\omega(\phi(n))$	170

§ V.20 Function $\omega(n; u, v)$	171
§ V.21 On the number of values $n \leq x$ with $\omega(n) > f(x)$	172
§ V.22 On $\omega(2^p - 1)$, $\Omega(a^n - 1)/n$	172
§ V.23 ω -highly composite, ω -largely composite and ω -interesting numbers	173
§ V.24 On $\omega(n)/n$	173
§ V.25 On $(n, \omega(n)) = 1$ and $(n, \Omega(n)) = 1$	174
§ V.26 On $\omega((n, \varphi(n))) = k$	174
§ V.27 Gaussian law of errors for ω	175
§ V.28 On the statistical property of prime factors of natural numbers in arithmetic progressions	176
§ V.29 Distribution of values of ω in short intervals	177
§ V.30 Distribution of ω in the sieve of Eratosthenes	177
§ V.31 Number of $n \leq x$ with $\Omega(n) = i$	177
§ V.32 Number of $n \leq x$ with $\omega(n) = i$	180
§ V.33 The functions $\omega(n; E)$ and $S(x, y; E, \omega)$	183
§ V.34 Sumsets with many prime factors	184
§ V.35 On the integers n for which $\Omega(n) = k$	185

CHAPTER VI

FUNCTION μ ; k -FREE AND k -FULL NUMBERS	187
§ VI. 1 Average order of $\mu(n)$	187
§ VI. 2 Estimates for $M(x)$. Mertens' conjecture	187
§ VI. 3 μ in short intervals	189
§ VI. 4 Sums involving $\mu(n)$ with $p(n) > y$ or $P(n) < y$, $n \leq x$. Squarefree numbers with restricted prime factors	189
§ VI. 5 Oscillatory properties of $M(x)$ and related results	190
§ VI. 6 The function $M(n, T) = \sum_{d n, d \leq T} \mu(n)$	192
§ VI. 7 Möbius function of order k	193
§ VI. 8 Sums on $\mu(n)/n$, $\mu(n)/n^2$, $\mu^2(n)/n$	194
§ VI. 9 Sums on $\mu(n) \log n/n$, $\mu(n) \log n/n^2$	195
§ VI.10 Selberg's formula	196
§ VI.11 A sum on $\mu(n) \left[\frac{x}{n} \right]$	197
§ VI.12 A sum on $\mu(n)f(n)/n$, f -multiplicative, $0 \leq f(p) \leq 1$	197
§ VI.13 Gandhi's formula	197
§ VI.14 An extremal property of μ	198
§ VI.15 On a sum connected with the Möbius function	199
§ VI.16 Sums over $\frac{\mu^2(n)}{\omega(n)}$, $\frac{\mu^2(n)}{\omega^2(n)}$, $\frac{\mu^2(n)}{\varphi(n)}$, $\frac{\mu(n)}{nd(n)}$	199

§ VI.17 The distribution of integers having a given number of prime factors	200
§ VI.18 Number of squarefree integers $\leq x$	201
§ VI.19 On squarefree integers	202
§ VI.20 Intervals containing a squarefree integer	202
§ VI.21 Distribution of squarefree numbers	204
§ VI.22 On the frequency of pairs of squarefree numbers	205
§ VI.23 Smallest squarefree integer in an arithmetic progression	206
§ VI.24 The greatest squarefree divisor of n	208
§ VI.25 Estimates involving the greatest squarefree divisor of n	209
§ VI.26 Estimates for $N(x, y) = \text{card} \{n \leq x : \gamma(n) \leq y\}$	210
§ VI.27 Number of non-squarefree odd, positive integers $\leq x$	210
§ VI.28 Number of squarefree numbers $\leq X$ which are quadratic residues $(\bmod p)$	211
§ VI.29 Squarefree integers in nonlinear sequences	211
§ VI.30 Sumsets containing squarefree and k -free integers	212
§ VI.31 On the Möbius function	213
§ VI.32 Number of k -free integers $\leq x$	213
§ VI.33 Number of k -free integers $\leq x$, which are relatively prime to n	216
§ VI.34 Schnirelmann density of the k -free integers	217
§ VI.35 Powerfree integers represented by linear forms	218
§ VI.36 On the power-free value of a polynomial	218
§ VI.37 Number of r -free integers $\leq x$ that are in arithmetic progression	220
§ VI.38 Squarefree numbers as sums of two squares	221
§ VI.39 Distribution of unitary k -free integers	221
§ VI.40 Counting function of the (k, r) -integers	222
§ VI.41 Asymptotic formulae for powerful numbers	222
§ VI.42 Maximal k -full divisor of an integer	226
§ VI.43 Number of squarefull integers between successive squares	226

CHAPTER VII

FUNCTIONS $\pi(x)$, $\psi(x)$, $\theta(x)$, AND THE SEQUENCE OF PRIME NUMBERS 227

§ VII. 1 Estimates on $\pi(x)$. Chebyshev's theorem. The prime number theorem	227
§ VII. 2 Approximation of $\pi(x)$ by $\int_2^x \frac{dy}{\log y}$	228
§ VII. 3 On $\pi(x) - \text{li } x$. Sign changes	229
§ VII. 4 On $\pi(x) - \pi(x - y)$ for $y = x^\theta$	232
§ VII. 5 On $\pi(x + y) \leq \pi(x) + \pi(y)$	235
§ VII. 6 On $\sum_{q \leq k \leq n} (\pi^*(k) - \pi(k))$	237

§ VII. 7 A sum on $\frac{1}{\pi(n)}$	238
§ VII. 8 Number of primes $p \leq x$ for which $p+k$ is a prime and related questions	238
§ VII. 9 Number of primes $p \leq x$ with $\omega(p+2) \leq 2$	240
§ VII.10 Almost primes P_2 in intervals	240
§ VII.11 P_{21} in short intervals	241
§ VII.12 Consecutive almost primes	242
§ VII.13 Primes in short intervals	243
§ VII.14 Primes between x and $a \cdot x$, ($a > 1$, constant.) Bertrand's postulate	243
§ VII.15 On intervals containing no primes	245
§ VII.16 Difference between consecutive primes	245
§ VII.17 Comparison of $p_1 \dots p_n$ with p_{n+1}	246
§ VII.18 Elementary estimates on $p_{[an]}, p_{mn}, p_{n+1}/p_n$	247
§ VII.19 Sharp upper and lower bounds for p_n	247
§ VII.20 The n th composite number	247
§ VII.21 On infinite series involving $\sqrt{p_{n+1}} - \sqrt{p_n}$, $1/n(p_{n+1} - p_n)$ and related problems	248
§ VII.22 Largest gap between consecutive primes below x	249
§ VII.23 On $\min(d_n, d_{n+1})$ and various sums over d_n	250
§ VII.24 On the sign changes of $d_n - d_{n+1}$ and related theorems on primes	253
§ VII.25 The sequence (b_n) defined by $b_n = d_n / \log p_n$	254
§ VII.26 Results on p_k/k	256
§ VII.27 On the sums of prime powers	257
§ VII.28 Estimates on $\sum_{p \leq x} \frac{1}{p}$	257
§ VII.29 Estimates on $\prod_{p \leq x} \left(1 - \frac{1}{p}\right)$	259
§ VII.30 Some properties of ψ -function	259
§ VII.31 Selberg's formula	262
§ VII.32 On $\sum_{n \leq x} \psi(n)$	263
§ VII.33 Estimates on $\psi(x+h) - \psi(x)$	263
§ VII.34 On $\Delta(x) = \psi(x) - x$	264
§ VII.35 Results on $\theta(x)$	267
§ VII.36 Primes in short intervals	270
§ VII.37 Estimates concerning $\Lambda(n)$ and certain generalizations. Sign-changes in the remainder	270
§ VII.38 A sum over $1/\Lambda(n)$	273
§ VII.39 On Chebyshev's conjecture	273

§ VII.40 A sum involving primes	274
---	-----

CHAPTER VIII

PRIMES IN ARITHMETIC PROGRESSIONS AND OTHER SEQUENCES	275
§ VIII. 1 Dirichlet's theorem on arithmetic progressions	275
§ VIII. 2 Bertrand's and related problems in arithmetic progressions	275
§ VIII. 3 Sums over $1/p$, $\log p/p$ when $p \leq x$, $p \equiv l(\bmod k)$	276
§ VIII. 4 The n -th prime in an arithmetic progression	278
§ VIII. 5 Least prime in an arithmetic progression. Linnik's theorem.	
Various estimates on $p(k, l)$	278
§ VIII. 6 Siegel-Walfisz theorem. The Bombieri-Vinogradov theorem . .	280
§ VIII. 7 Primes in arithmetic progressions	283
§ VIII. 8 Bombieri's theorem in short intervals	283
§ VIII. 9 Prime number theorem for arithmetic progressions	285
§ VIII.10 An estimate on $\pi(x; p, -1)$	285
§ VIII.11 Assertions equivalent to the prime number theorem for arithmetic progressions. Sums over $\pi(x; k, l) - \frac{\text{li } x}{\varphi(k)}$	286
§ VIII.12 Brun-Titchmarsh theorem	287
§ VIII.13 Application of the Brun-Titchmarsh theorem on lower bounds for $\pi(x; k, l)$	290
§ VIII.14 On $\pi(x + x^\theta; k \cdot l) - \pi(x; k, l)$	290
§ VIII.15 Barban's theorem	291
§ VIII.16 On generalizations of the Bombieri-Vinogradov theorem	291
§ VIII.17 An upper bound for $\pi_k(y; k, l) = \text{number of primes}$ $x < p \leq x + y$ with $p \equiv l(\bmod k)$	292
§ VIII.18 An analogue of the Brun-Titchmarsh inequality	292
§ VIII.19 On Goldbach-Vinogradov's theorem. The prime k -tuple conjecture on average	293
§ VIII.20 Sums over $\left(\theta(x; k, l) - \frac{x}{\varphi(k)} \right)^2$, $\left \pi(x; k, l) - \frac{\text{li } x}{\varphi(k)} \right ^2$. .	294
§ VIII.21 Oscillation theorems for primes in arithmetic progressions . .	295
§ VIII.22 Special results on finite sums over primes	297
§ VIII.23 Infinitely many sets of three distinct primes and an almost prime in arithmetic progressions	297
§ VIII.24 Large prime factors of integers in an arithmetic progression .	298
§ VIII.25 Almost primes in arithmetic progressions	299
§ VIII.26 Arithmetic progressions that consist only in primes	299
§ VIII.27 Number of $n \leq x$ such that there is no prime between n^2 and $(n+1)^2$	299
§ VIII.28 Primes in the sequence $[n^c]$	300
§ VIII.29 Number of primes $p \leq x$ for which $[p^c]$ is prime	301
§ VIII.30 Almost primes in $(n^2 + 1)$ and related sequences	302

§ VIII.31 Primes $p \leq N$ of the form $p = [cn]$	304
§ VIII.32 Primes of the form $n \cdot 2^n + 1$ or $p \cdot 2^p + 1$ or $2^p \pm p$	305
§ VIII.33 Primes of the form $x^2 + y^2 + 1$	306
§ VIII.34 On a sum on $\frac{\log p}{p}$ when $p \in L$ ≠ arithmetic progression	306
§ VIII.35 Recurrent sequences of primes	307
§ VIII.36 Composite values of exponential and related sequences	307
§ VIII.37 Primes in partial sums of n^n	308
§ VIII.38 Beurling's generalized integers	308
§ VIII.39 Accumulation theorems for primes in arithmetic progressions	309
§ VIII.40 About the Shanks-Rényi race problem	311

CHAPTER IX

ADDITIVE AND DIOPHANTINE PROBLEMS INVOLVING PRIMES	313
§ IX. 1 Schnirelman's theorem. Vinogradov's theorem	313
§ IX. 2 Number of representations of N in the form $p_1^n + \dots + p_k^n$. Vinogradov's three primes theorem	314
§ IX. 3 Rényi's theorem. Chen's theorem	316
§ IX. 4 Improvements on Chen's theorem	317
§ IX. 5 On number of writings of N as $\varepsilon_1 \dots \varepsilon_s + p_1 \dots p_r$, or $\varepsilon_1 \dots \varepsilon_s + p_1 \dots p_{r+1}$. A common generalization of Chen's and Linnik's theorems	318
§ IX. 6 On $p_1^k + p_2^k = N$. Estimates on the number of solutions. Binary Hardy-Littlewood problem	320
§ IX. 7 Number of Goldbach numbers and related problems	321
§ IX. 8 The exceptional set in Goldbach's problem	323
§ IX. 9 Partitions into primes	324
§ IX.10 Partitions of n into parts, or distinct parts in a set A	326
§ IX.11 Representations in the form $k = ap_1 + \dots + a_r p_r$, (p_i primes) with restricted primes p_i	327
§ IX.12 Representations in the form $N = p + n$, p prime, with certain restrictions on n	327
§ IX.13 On integers of the form $p + a^k$ (p prime, $a > 1$) or $p^2 + a^k$ or $p + q!$ (q prime), etc.	328
§ IX.14 Linnik's theorem (on the Hardy-Littlewood problem)	330
§ IX.15 Representations in the form $p_1^3 + p_2^3 + p_3^3 + x^3$ (p_i primes), etc.	332
§ IX.16 Number of solutions of $n = p + xy$ (p prime; $x, y \geq 1$)	332
§ IX.17 Representations of primes by quadratic forms	333

§ IX.18 Number of solutions of $m = p_1 + v^a$, $n = p_2 + v^a$, ($m < x$, $n < x$, p_i primes)	333
§ IX.19 Number of representations of n as the sum of the square of a prime and an r -free integer	334
§ IX.20 Distinct integers $\leq x$ which can be expressed as $p + a^{k_i}$, where (k_i) is a certain sequence	334
§ IX.21 Waring-Goldbach-type problems for the function $f(x) = x^c$, $c > 12$. Hybrid of theorems by Vinogradov and Pjateckii-Šapiro	335
§ IX.22 Integers not representable in the form $p + [n^c]$ ($c > 1$)	336
§ IX.23 On the maximal distance between integers composed of small primes	336
§ IX.24 On the representation of N as $N = a + b$ or $N = a + b + c$ with restrictions on $P(ab)$ or $P(abc)$	337
§ IX.25 On the maximal length of two sequences of consecutive integers with the same prime divisors	339
§ IX.26 Representation of n as $n = \frac{p+1}{q+1}$ (p, q primes)	339
§ IX.27 An additive property of squares and primes	341
§ IX.28 On the distribution of $\{\sqrt{p}\}$ and $\{p^\theta\}$, $\frac{1}{2} \leq \theta \leq 1$	342
§ IX.29 Diophantine approximations by almost primes	343
§ IX.30 Number of solutions of $\ f(p)\ < p^{-\epsilon+\epsilon}$ (p prime)	343
§ IX.31 A sum involving $\ p\alpha\ $ (p prime)	344
§ IX.32 On the distribution of αp modulo one	344
§ IX.33 Simultaneous diophantine approximation with primes	345
§ IX.34 Diophantine approximation by prime numbers	346
§ IX.35 Metric diophantine approximation with two restricted prime variables	347
§ IX.36 The uniform distributed sequences (αp) and (p^α) , where $0 < \alpha < 1$, and (p^γ) , $\gamma > 1$, $\gamma \neq$ integer	348

CHAPTER X

EXPONENTIAL SUMS

§ X. 1 Basic estimates on $\sum_{n \leq x} e(m\alpha)$	349
§ X. 2 Weyl's method	349
§ X. 3 van der Corput's method	350
§ X. 4 Vinogradov's method	353
§ X. 5 Theory of exponent pairs	353
§ X. 6 Multiple trigonometric sums	355
§ X. 7 Estimates on $\int_c^b g(t) \cdot e^{if(t)} dt$	356

§ X. 8 Estimates of type $\iint_D e^{if(x,y)} dx dy$ or $\sum_{(n,m) \in D} e(f(n,m))$	
where D is a plane domain	357
§ X. 9 Vinogradov's mean-value theorem	359
§ X.10 Exponential sums containing primes	360
§ X.11 Exponential sums of type $\sum_{M' \leq m \leq M} (m+w)^{ti}$	361
§ X.12 Complete trigonometric sums	362
§ X.13 Nearly complete and supercomplete rational trigonometric sums	364
§ X.14 Hua's estimate	365
§ X.15 Gaussian sums	366
§ X.16 Estimates by Linnik and Vinogradov	366
§ X.17 Sums of type $\sum_{p \leq N} (\log p) \cdot e(ap^k/q)$ (p prime) and $\sum_{p \leq N} e(p\alpha)$ where $\left \alpha - \frac{a}{q} \right \leq \frac{1}{q^2}$ for $(a, q) = 1$	367
§ X.18 Estimates of trigonometric sums over primes in short intervals	369
§ X.19 A short exponential rational trigonometric sum	371
§ X.20 Estimates on sums over $e(uh/k)$, when $f(u) \equiv 0 \pmod{k}$, $0 < u \leq k$ and $k \leq x$	372
§ X.21 Exponential sums formed with the Möbius function	372
§ X.22 On $\sum_{n \leq x} \mu^2(n) e(\alpha n^3)$	373
§ X.23 The sum of $e(\alpha n)$, when $\Omega(n) = k$	374
§ X.24 Exponential sums involving the Ramanujan function	374
§ X.25 An exponential sum involving $r(n)$ (number of representations of n as a sum of two squares)	375
§ X.26 Exponential sums on integers having small prime factors	375
§ X.27 A result on $\sum_{n \leq N} e(x\sqrt{n})$	376
§ X.28 Kloosterman sums. Salié's and Weil's estimates	377
§ X.29 Exponential sums connected with the distribution of $\alpha p \pmod{1}$ and with diophantine approximation with primes or almost primes	378
§ X.30 On $e(\alpha x^3)$	379
§ X.31 Exponential sums and the logarithmic uniform distribution of $(\alpha n + \beta \log n)$	380
§ X.32 Exponential sums with multiplicative coefficients	381
§ X.33 On $\sum \Phi(u) \Psi(v) e(f(uv))$	382
§ X.34 Exponential sums involving quadratic polynomials and sequences	383
§ X.35 The large sieve as an estimate for exponential sums	383
§ X.36 An estimate for the derivative of a trigonometric polynomial	386

§ X.37 Weighted exponential sums and discrepancy	386
§ X.38 Deligne's estimates	386
§ X.39 On fourth moments of exponential sums	387
§ X.40 Biquadratic Weyl sums	387
CHAPTER XI	
CHARACTER SUMS	389
§ XI. 1 Pólya-Vinogradov inequality and a generalization. Character sums modulo a prime power. Burgess' estimate	389
§ XI. 2 On the constant in the Pólya-Vinogradov inequality. Large values of character sums	390
§ XI. 3 Burgess' character sum estimate	393
§ XI. 4 A character sum estimate for nonprincipal character $\chi(\text{mod } q)$	393
§ XI. 5 A sum on $\chi(u+v)$, on sets with no two integers of which are congruent	394
§ XI. 6 A lower bound on a character sum estimate arising in a problem concerning the distribution of sequences of integers in arithmetic progressions	394
§ XI. 7 Powers of character sums	394
§ XI. 8 Sums of characters with primes. Vinogradov's theorem	396
§ XI. 9 Distribution of pairs of residues and nonresidues of special form	397
§ XI.10 A character sum estimate involving $\mu(n)$ and $\Lambda(n)$	397
§ XI.11 An upper bound for a character sum involving $\Lambda(n)$	398
§ XI.12 Half Gauss sums	398
§ XI.13 Exponential sums with characters. A large-sieve density estimate	399
§ XI.14 On $\sum_{k=1}^{q-1} \chi(n) \cdot k^n$	400
§ XI.15 Estimates on $\sum_{x=N+1}^M \chi(x) \cdot e(ax/p)$	401
§ XI.16 An infinite series of characters with application to zero density estimates for \mathcal{L} functions	402
§ XI.17 Character sums of polynomials	402
§ XI.18 Quadratic character of a polynomial	403
§ XI.19 Distribution of values of characters in sparse sequence	404
§ XI.20 Estimation of character sums modulo a power of a prime	404
§ XI.21 Mean values of character sums	406
§ XI.22 On $\sum_{n \in S(x,y)} \chi(n)$, with $S(x,y) = \{ n \leq x : P(n) \leq y \}$	407
§ XI.23 Large sieve-type inequalities via character sum estimates	407
§ XI.24 Large sieve-type inequalities of Selberg and Motohashi	409
§ XI.25 A large sieve density estimate	410
§ XI.26 A theorem by Wolke	410

§ XI.27 Character sums involving $\Psi(X, \chi) = \sum_{n \leq X} \Lambda(n) \chi(n)$	411
§ XI.28 An estimate involving $\chi_1 * \chi_2$	411
§ XI.29 Number of primitive characters mod n , and the number of characters with modulus $\leq x$	412
§ XI.30 Continuous additive characters of a topological abelian group	413
§ XI.31 An estimate for perturbed Dirichlet characters	413
§ XI.32 Estimates on Hecke characters	413
§ XI.33 Character sums in finite fields	414
§ XI.34 Gauss sums, Kloosterman sums	415
§ XI.35 Dirichlet characters on additive sequences	416
CHAPTER XII	
BINOMIAL COEFFICIENTS, CONSECUTIVE INTEGERS AND RELATED PROBLEMS	417
§ XII. 1 On $p^a \mid \binom{n}{k}$	417
§ XII. 2 Number of binomial coefficients not divisible by an integer	418
§ XII. 3 Number of distinct prime factors of binomial coefficients	419
§ XII. 4 Divisibility properties of $\binom{2n}{n}$	422
§ XII. 5 Squarefree divisors of $\binom{2n}{n}$	424
§ XII. 6 Divisibility properties of consecutive integers	425
§ XII. 7 The theorem of Sylvester and Schur	426
§ XII. 8 On the prime factorization of binomial coefficients	427
§ XII. 9 Inequalities and estimates involving binomial coefficients	430
§ XII.10 On unimodal sequences of binomial coefficients	434
§ XII.11 A theorem of Pillai and Szekeres	435
§ XII.12 A sum on a function connected with consecutive integers	436
§ XII.13 On consecutive integers. Theorems of Erdős-Rankin and Shorey	436
§ XII.14 On prime factors on consecutive integers	437
§ XII.15 The Grimm conjecture and analogues problems	438
§ XII.16 Great values of a function connected with consecutive integers	440
§ XII.17 A theorem of Erdős and Selfridge on the product of consecu- tive integers	440
§ XII.18 Products terms in an arithmetical progression	441
§ XII.19 On the sequence $n! + k$, $2 \leq k \leq n$	442
§ XII.20 Decomposition of $n!$ into prime factors	442
§ XII.21 Divisibility of products of factorials	444
§ XII.22 Powers and factorials	445
§ XII.23 Distribution of divisors of $n!$	447
§ XII.24 Stirling's formula and power of factorials	447

§ XII.25 The Wallis sequence and related inequalities on gamma function	448
§ XII.26 A special sequence of Cesáro	450
§ XII.27 Inequalities on powers and factorials related to the gamma function	451
§ XII.28 Arithmetical products involving the gamma function	451
§ XII.29 Monotonicity and convexity results of certain expressions of gamma function	452
§ XII.30 Left factorial function	457

CHAPTER XIII

ESTIMATES INVOLVING FINITE GROUPS AND SEMI-SIMPLE RINGS	459
§ XIII. 1 Maximal order of an element in the symmetric group	459
§ XIII. 2 A sum on the order of elements of S_n	460
§ XIII. 3 Statistical problems in S_n	461
§ XIII. 4 Probability of generating the symmetric group	462
§ XIII. 5 Primitive subgroups of S_n	463
§ XIII. 6 Number of solutions of $x^k = 1$ in symmetric groups	464
§ XIII. 7 On the dimensions of representations of S_n	465
§ XIII. 8 Conjugacy classes of the alternating group of degree n	466
§ XIII. 9 An estimate for the order of rational matrices	467
§ XIII.10 On k th power coset representatives mod p	467
§ XIII.11 Arithmetical properties of permutations of integers	467
§ XIII.12 Number of non-isomorphic abelian groups of order n	468
§ XIII.13 Abelian groups of a given order	472
§ XIII.14 Number of non-isomorphic abelian groups in short intervals	472
§ XIII.15 Number of representations of n as a product of k -full numbers	473
§ XIII.16 Number of distinct values taken by $a(n)$ and related problems	474
§ XIII.17 Number of $n \leq x$ with $a(n) = a(n+1)$. The functions $a(n)$ at consecutive integers	475
§ XIII.18 Sums involving $(\Omega(n+1) - \omega(n+1)) \cdot a(n)$, $d(n+1)a(n)$, $\omega(n+1)a(n)$	476
§ XIII.19 On sums involving $\frac{1}{a(n)}$ and $\frac{1}{\log a(n)}$	477
§ XIII.20 The iterates of $a(n)$	477
§ XIII.21 Statistical theorems on the embedding of abelian groups into symmetrical ones	478
§ XIII.22 Probabilistic results in group theory	479
§ XIII.23 Finite abelian group cohesion	480
§ XIII.24 Number of non-isomorphic groups of order n	481
§ XIII.25 Density of finite simple group orders	483
§ XIII.26 Large cyclic subgroups of finite groups	484

§ XIII.27 Counting solvable, cyclic, nilpotent groups orders	484
§ XIII.28 On C -groups	485
§ XIII.29 The order of directly indecomposable groups. Direct factors of a finite abelian groups	486
§ XIII.30 On a family of almost cyclic finite groups	487
§ XIII.31 Asymptotic results for elements of a semigroup	488
§ XIII.32 Number of non-isomorphic semi-simple finite rings of order n	489
§ XIII.33 On a problem of Rohrbach for finite groups	490
§ XIII.34 On cocyclicity of finite groups	490

CHAPTER XIV

PARTITIONS	491
§ XIV. 1 Unrestricted partitions of an integer	491
§ XIV. 2 Partitions of n into exactly k positive parts	493
§ XIV. 3 Partitions of n into at most k summands	495
§ XIV. 4 Unequal partitions of n containing each a_i as a summand . .	497
§ XIV. 5 Partitions of n into members of a finite set	498
§ XIV. 6 Partitions of n without a given subsum	498
§ XIV. 7 Partitions of n which no part is repeated more than t times .	499
§ XIV. 8 Partitions of n whose parts are $\geq m$	499
§ XIV. 9 Partitions of n into unequal parts $\geq m$	501
§ XIV. 10 On the subsums of a partition	502
§ XIV. 11 On other subsums of a partition	504
§ XIV. 12 Partitions of j -partite numbers into k summands	505
§ XIV. 13 On a result of Turán	507
§ XIV. 14 Statistical theory of partitions	507
§ XIV. 15 Partitions of n into distinct parts all $\equiv a_i \pmod{m}$	508
§ XIV. 16 Partitions with congruences conditions	508
§ XIV. 17 Partitions of n whose parts are relatively prime, or prime to n , etc.	509
§ XIV. 18 Partitions of n whose parts a_i ($i = 1, k$) satisfy $a_1 a_2 \dots a_k$	510
§ XIV. 19 Partitions of n as sums of powers of 2	512
§ XIV. 20 Partitions of n into powers of r (≥ 2)	512
§ XIV. 21 On a problem of Frobenius	513
§ XIV. 22 An Abel-Tauber problem for partitions	514
§ XIV. 23 On partitions of the positive integers with no x, y, z belonging to distinct classes satifying $x + y = z$	515
§ XIV. 24 On certain partitions of n into $r \geq 2$ distinct pairs	515
§ XIV. 25 Additively independent partitions	516
§ XIV. 26 A problem in “factorisatio numerorum” of Kalmár	516
§ XIV. 27 Cyclotomic partitions	519

§ XIV. 28 Multiplicative properties of the partition function	520
§ XIV. 29 Partitions into primes	520
§ XIV. 30 Partitions of N into terms of $1, 2, \dots, n$, repeating a term at most p times	520
§ XIV. 31 Partition which assumes all integral values	521
§ XIV. 32 Partitions free of small summands	521
CHAPTER XV	
CONGRUENCES, RESIDUES AND PRIMITIVE ROOTS	523
§ XV. 1 Addition of residue classes $\text{mod } p$	523
§ XV. 2 Residues of n^n	524
§ XV. 3 Distribution of quadratic nonresidues	524
§ XV. 4 Distribution of quadratic residues	526
§ XV. 5 Sequences of consecutive quadratic nonresidues	528
§ XV. 6 On residue difference sets	529
§ XV. 7 Sets which contain a quadratic residue $\text{mod } p$ for almost all p	530
§ XV. 8 Least prime quadratic residue	530
§ XV. 9 Quadratic residues of squarefree integers	530
§ XV. 10 Least k -th power nonresidue	531
§ XV. 11 Quadratic residues in arithmetic progressions	532
§ XV. 12 Bounds on n -th power residues ($\text{mod } p$)	534
§ XV. 13 Positive d -th power residues $\leq x$, with $d (p - 1)$, which are prime to A	534
§ XV. 14 Distribution of r -th powers in a finite field	534
§ XV. 15 Pólya-Vinogradov inequality for quadratic characters	535
§ XV. 16 Distribution questions concerning the Legendre symbol	535
§ XV. 17 A sum on $\left(\frac{n}{p}\right) \cdot n^k$	536
§ XV. 18 An exponential polynomial formed with the Legendre symbol	537
§ XV. 19 A mean value of a quadratic character sum	537
§ XV. 20 Two sums involving Legendre's symbol with primes	537
§ XV. 21 Least primitive roots $\text{mod } p$. Least primitive roots $\text{mod } p^2$. Number of solutions of congruence $x^{n-1} \equiv 1 \pmod{n}$ for n composite	538
§ XV. 22 Distribution of primitive roots of a prime	541
§ XV. 23 Artin's conjecture on primitive roots	542
§ XV. 24 Number of primitive roots $\leq x$ which are $\equiv 1 \pmod{k}$	543
§ XV. 25 Number of squarefull (squarefree) primitive roots $\leq x$	543
§ XV. 26 Number of integers in $[M + 1, M + N]$ which are not primitive roots ($\text{mod } p$) for any $p \leq N^{1/2}$	544
§ XV. 27 Least prime primitive roots	544
§ XV. 28 Fibonacci primitive roots	545

§ XV.29 Distribution of primitive roots in finite fields	545
§ XV.30 Number of solutions to $f(x) \equiv 0 \pmod{m}$ counted \pmod{m}	545
§ XV.31 Estimates on Legendre symbols of polynomials	547
§ XV.32 Number of solutions to $f(x) \equiv a \pmod{p^b}$ (p prime)	548
§ XV.33 Number of residue classes $k \pmod{r}$ with $f(k) \equiv 0 \pmod{r}$	549
§ XV.34 Zeros of polynomials over finite fields	550
§ XV.35 Congruences on homogenous linear forms	552
§ XV.36 Waring's problem (\pmod{p})	553
§ XV.37 Estimate of Mordell on congruences	553
§ XV.38 Distribution of solutions of congruences	554
§ XV.39 On a set of congruences related to character sums	555
§ XV.40 Small zeros of quadratic congruences \pmod{p}	555
§ XV.41 Congruence-preserving arithmetical functions	556
§ XV.42 On a congruence of Mirimanoff type	556

CHAPTER XVI

ADDITIVE AND MULTIPLICATIVE FUNCTIONS	557
§ XVI. 1 Erdős' theorem on additive functions with difference tending to zero, generalizations, extensions and related results	557
§ XVI. 2 Completely additive functions with restricted growth	560
§ XVI. 3 Turán-Kubilius inequality	561
§ XVI. 4 Erdős-Kac theorem	563
§ XVI. 5 Erdős-Wintner theorem	564
§ XVI. 6 Value distribution of differences of additive functions	566
§ XVI. 7 Erdős-Wintner theorem for normed semigroups	567
§ XVI. 8 Turán-Kubilius inequality and the Erdős-Wintner theorem for additive functions of a rational argument	567
§ XVI. 9 Limit theorem for additive functions on ordered semigroups . .	568
§ XVI.10 Laws of iterated logarithm for additive functions	569
§ XVI.11 Limit laws and moments of additive functions in short intervals	570
§ XVI.12 Distribution function of the sum of an additive and multiplicative function	571
§ XVI.13 Moments and concentration of additive functions	571
§ XVI.14 Local theorems for additive functions	572
§ XVI.15 Additive functions on arithmetic progressions	574
§ XVI.16 On differences of additive functions	575
§ XVI.17 Prime-independent additive functions	577
§ XVI.18 Moments and Cesàro means of additive functions	577
§ XVI.19 Minimax-theorem for additive functions	579
§ XVI.20 Maximal value of additive functions in short intervals	580
§ XVI.21 Normal order of additive functions on sets of shifted primes .	581
§ XVI.22 Uniformly distributed $(\pmod{1})$ additive functions	582
§ XVI.23 Additive functions and almost periodicity	582
§ XVI.24 Characterization of multiplicative functions	582
§ XVI.25 Multiplicative functions with small increments	583

§ XVI.26 Conditions on a multiplicative function to be completely multiplicative	584
§ XVI.27 Delange's theorem on mean-values of multiplicative functions	584
§ XVI.28 Halász' theorem	587
§ XVI.29 Wirsing's theorem	588
§ XVI.30 Mean value of fg and $f * g$	590
§ XVI.31 Mean value of $f(P(n))$, P a polynomial	591
§ XVI.32 Multiplicative functions $ f \leq 1$: Summation formulas	591
§ XVI.33 Indlekofer's theorem	592
§ XVI.34 Cesàro means of additive functions	593
§ XVI.35 Multiplicative functions on short intervals	594
§ XVI.36 Multiplicative functions on arithmetic progressions. Elliott's theorems	595
§ XVI.37 Effective mean value estimate for complex multiplicative functions	597
§ XVI.38 A theorem of Levin, Timofeev and Tuliagonov on the distribution of multiplicative functions. The Bakshtys-Galambos theorems	599
§ XVI.39 Sums on multiplicative functions satisfying certain conditions	600
§ XVI.40 An asymptotic summation formula for multiplicative functions with $ f(n) \leq 1$	601
§ XVI.41 An Ω -estimate for the remainder of sums of multiplicative functions	601
§ XVI.42 The distribution of values of some multiplicative functions	602
§ XVI.43 Multiplicative functions and small divisors	603
§ XVI.44 An estimate for submultiplicative functions	604
§ XVI.45 Divisibility properties of some multiplicative functions	604
§ XVI.46 On multiplicative functions satisfying a congruence relation	605
§ XVI.47 Exponential sums with multiplicative function coefficients	605
§ XVI.48 Ramanujan expansions of multiplicative functions	606
§ XVI.49 Asymptotic formulae for reciprocals of quotients of additive and multiplicative functions	606
§ XVI.50 Semigroup-valued multiplicative functions	609
 INDEX OF AUTHORS	611