

# Contents

<b>Preface</b>	<b>xi</b>
<b>0 Introduction</b>	<b>1</b>
0.1 Tree lattices . . . . .	1
0.2 $X$ -lattices and $H$ -lattices . . . . .	2
0.3 Near simplicity . . . . .	3
0.4 The structure of tree lattices . . . . .	4
0.5 Existence of lattices . . . . .	4
0.6 The structure of $A = \Gamma \backslash X$ . . . . .	6
0.7 Volumes . . . . .	7
0.8 Centralizers, normalizers, commensurators . . . . .	8
<b>1 Lattices and Volumes</b>	<b>13</b>
1.1 Haar measure . . . . .	13
1.2 Lattices and unimodularity . . . . .	13
1.3 Compact open subgroups . . . . .	13
1.5 Discrete group covolumes . . . . .	14
<b>2 Graphs of Groups and Edge-Indexed Graphs</b>	<b>17</b>
2.1 Graphs . . . . .	17
2.2 Morphisms and actions . . . . .	18
2.3 Graphs of groups . . . . .	18
2.4 Quotient graphs of groups . . . . .	19
2.5 Edge-indexed graphs and their groupings . . . . .	19
2.6 Unimodularity, volumes, bounded denominators . . . . .	20
<b>3 Tree Lattices</b>	<b>25</b>
3.1 Topology on $G = \text{Aut}X$ . . . . .	25
3.2 Tree lattices . . . . .	25
3.3 The group $G_H$ of deck transformations . . . . .	26
3.5 Discreteness Criterion; Rigidity of $(A, i)$ . . . . .	27

3.6	Unimodularity and volume . . . . .	29
3.8	Existence of tree lattices . . . . .	30
3.12	The structure of tree lattices . . . . .	31
3.14	Non-arithmetic uniform commensurators . . . . .	33
<b>4</b>	<b>Arbitrary Real Volumes, Cusps, and Homology</b>	<b>35</b>
4.0	Introduction . . . . .	35
4.1	Grafting . . . . .	36
4.2	Volumes . . . . .	38
4.8	Cusps . . . . .	43
4.9	Geometric parabolic ends . . . . .	44
4.10	$\Gamma$ -parabolic ends and $\Gamma$ -cusps . . . . .	49
4.11	Unidirectional examples . . . . .	51
4.12	A planar example . . . . .	54
<b>5</b>	<b>Length Functions, Minimality</b>	<b>67</b>
5.1	Hyperbolic length (cf. [B3], II, §6) . . . . .	67
5.4	Minimality . . . . .	68
5.14	Abelian actions . . . . .	71
5.15	Non-abelian actions . . . . .	71
5.16	Abelian discrete actions . . . . .	71
<b>6</b>	<b>Centralizers, Normalizers, and Commensurators</b>	<b>73</b>
6.0	Introduction . . . . .	73
6.1	Notation . . . . .	74
6.6	Non-minimal centralizers . . . . .	77
6.9	$N/\Gamma$ , for minimal non-abelian actions . . . . .	80
6.10	Some normal subgroups . . . . .	81
6.11	The Tits Independence Condition . . . . .	82
6.13	Remarks . . . . .	85
6.16	Automorphism groups of rooted trees . . . . .	86
6.17	Automorphism groups of ended trees . . . . .	88
6.21	Remarks . . . . .	90
<b>7</b>	<b>Existence of Tree Lattices</b>	<b>91</b>
7.1	Introduction . . . . .	91
7.2	Open fanning . . . . .	93
7.5	Multiple open fanning . . . . .	98

<b>8</b>	<b>Non-Uniform Lattices on Uniform Trees</b>	<b>103</b>
8.1	Carbone's Theorem . . . . .	103
8.6	Proof of Theorem (8.2) . . . . .	110
8.7	Remarks . . . . .	110
8.8	Examples. Loops and cages . . . . .	111
8.9	Two vertex graphs . . . . .	116
<b>9</b>	<b>Parabolic Actions, Lattices, and Trees</b>	<b>119</b>
9.0	Introduction . . . . .	119
9.1	$Ends(X)$ . . . . .	120
9.2	Horospheres and horoballs . . . . .	121
9.3	End stabilizers . . . . .	122
9.4	Parabolic actions . . . . .	123
9.5	Parabolic trees . . . . .	125
9.6	Parabolic lattices . . . . .	125
9.8	Restriction to horoballs . . . . .	126
9.9	Parabolic lattices with linear quotient . . . . .	127
9.10	Parabolic ray lattices . . . . .	131
9.13	Parabolic lattices with all horospheres infinite . . . . .	139
9.14	A bounded degree example . . . . .	143
9.15	Tree lattices that are simple groups must be parabolic . . . . .	148
9.16	Lattices on a product of two trees . . . . .	149
<b>10</b>	<b>Lattices of Nagao Type</b>	<b>151</b>
10.1	Nagao rays . . . . .	151
10.2	Nagao's Theorem: $\Gamma = PGL_2(F_q[t])$ . . . . .	157
10.3	A divisible $(q + 1)$ -regular grouping . . . . .	160
10.4	The PNeumann groupings . . . . .	161
10.5	The symmetric groupings . . . . .	163
10.6	Product groupings . . . . .	164
	<b>Appendix [BCR]: The Existence Theorem for Tree Lattices</b>	
	<i>Hyman Bass, Lisa Carbone, and Gabriel Rosenberg</i>	<b>167</b>
	<b>Appendix [BT]: Discreteness Criteria for Tree Automorphism Groups</b>	
	<i>Hyman Bass and Jacques Tits</i>	<b>185</b>
	<b>Appendix [PN]: The PNeumann Groups</b>	
	<i>Hyman Bass and Alexander Lubotzky</i>	<b>213</b>
	<b>References</b>	<b>223</b>
	<b>Index of Notation</b>	<b>229</b>
	<b>Index</b>	<b>231</b>