

Table of Contents

Prerequisites	XIII
Notation	XIX
1. Introduction and Historical Survey	1
1.1 Liouville, Hermite, Lindemann, Gel'fond, Baker	1
1.2 Lower Bounds for $ a_1^{b_1} \cdots a_m^{b_m} - 1 $	6
1.3 The Six Exponentials Theorem and the Four Exponentials Conjecture	13
1.4 Algebraic Independence of Logarithms	15
1.5 Diophantine Approximation on Linear Algebraic Groups	19
Exercises	21

Part I. Transcendence

2. Transcendence Proofs in One Variable	29
2.1 Introduction to Transcendence Proofs	29
2.2 Auxiliary Lemmas	33
2.3 Schneider's Method with Alternants – Real Case	37
2.4 Gel'fond's Method with Interpolation Determinants – Real Case ..	43
2.5 Gel'fond-Schneider's Theorem in the Complex Case	49
2.6 Hermite-Lindemann's Theorem in the Complex Case	55
Exercises	59
3. Heights of Algebraic Numbers	65
3.1 Absolute Values on a Number Field	66
3.2 The Absolute Logarithmic Height (Weil)	75
3.3 Mahler's Measure	78
3.4 Usual Height and Size	80
3.5 Liouville's Inequalities	82
3.6 Lower Bound for the Height	86
Open Problems	105
Exercises	106

Appendix – Inequalities Between Different Heights of a Polynomial –
 From a Manuscript by Alain Durand 113

4. The Criterion of Schneider-Lang 115

4.1 Algebraic Values of Entire Functions Satisfying Differential
 Equations 115

4.2 First Proof of Baker’s Theorem 118

4.3 Schwarz’ Lemma for Cartesian Products 122

4.4 Exponential Polynomials 130

4.5 Construction of an Auxiliary Function 131

4.6 Direct Proof of Corollary 4.2 136

Exercises 141

Part II. Linear Independence of Logarithms and Measures

5. Zero Estimate, by Damien Roy 147

5.1 The Main Result 147

5.2 Some Algebraic Geometry 150

5.3 The Group G and its Algebraic Subgroups 156

5.4 Proof of the Main Result 164

Exercises 166

6. Linear Independence of Logarithms of Algebraic Numbers 169

6.1 Applying the Zero Estimate 170

6.2 Upper Bounds for Alternants in Several Variables 175

6.3 A Second Proof of Baker’s Homogeneous Theorem 181

Exercises 184

7. Homogeneous Measures of Linear Independence 187

7.1 Statement of the Measure 187

7.2 Lower Bound for a Zero Multiplicity 192

7.3 Upper Bound for the Arithmetic Determinant 195

7.4 Construction of a Nonzero Determinant 199

7.5 The Transcendence Argument — General Case 203

7.6 Proof of Theorem 7.1 — General Case 208

7.7 The Rational Case: Fel’dman’s Polynomials 214

7.8 Linear Dependence Relations between Logarithms 222

Open Problems 227

Exercises 227

Part III. Multiplicities in Higher Dimension

8. Multiplicity Estimates, by Damien Roy	231
8.1 The Main Result	231
8.2 Some Commutative Algebra	234
8.3 The Group G and its Invariant Derivations	238
8.4 Proof of the Main Result	245
Exercises	247
9. Refined Measures	251
9.1 Second Proof of Baker's Nonhomogeneous Theorem	252
9.2 Proof of Theorem 9.1	262
9.3 Value of $C(m)$	286
9.4 Corollaries	302
Exercises	314
10. On Baker's Method	317
10.1 Linear Independence of Logarithms of Algebraic Numbers	317
10.2 Baker's Method with Interpolation Determinants	329
10.3 Baker's Method with Auxiliary Function	356
10.4 The State of the Art	360
Exercises	371

Part IV. The Linear Subgroup Theorem

11. Points Whose Coordinates are Logarithms of Algebraic Numbers	375
11.1 Introduction	375
11.2 One Parameter Subgroups	379
11.3 Six Variants of the Main Result	381
11.4 Linear Independence of Logarithms	387
11.5 Complex Toruses	394
11.6 Linear Combinations of Logarithms with Algebraic Coefficients	398
11.7 Proof of the Linear Subgroup Theorem	404
Exercises	411
12. Lower Bounds for the Rank of Matrices	417
12.1 Entries are Linear Polynomials	418
12.2 Entries are Logarithms of Algebraic Numbers	432
12.3 Entries are Linear Combinations of Logarithms	435
12.4 Assuming the Conjecture on Algebraic Independence of Logarithms	437
12.5 Quadratic Relations	438
Exercises	441

Part V. Simultaneous Approximation of Values of the Exponential Function in Several Variables

13. A Quantitative Version of the Linear Subgroup Theorem	445
13.1 The Main Result	447
13.2 Analytic Estimates	450
13.3 Exponential Polynomials	459
13.4 Proof of Theorem 13.1	464
13.5 Directions for Use	471
13.6 Introducing Feld'man's Polynomials	476
13.7 Duality: the Fourier-Borel Transform	480
Exercises	490
14. Applications to Diophantine Approximation	495
14.1 A Quantitative Refinement to Gel'fond-Schneider's Theorem ...	496
14.2 A Quantitative Refinement to Hermite-Lindemann's Theorem ...	510
14.3 Simultaneous Approximation in Higher Dimension	521
14.4 Measures of Linear Independence of Logarithms (Again)	536
Open Problems	547
Exercises	549
15. Algebraic Independence	555
15.1 Criteria: Irrationality, Transcendence, Algebraic Independence ...	555
15.2 From Simultaneous Approximation to Algebraic Independence ...	569
15.3 Algebraic Independence Results: Small Transcendence Degree ...	587
15.4 Large Transcendence Degree: Conjecture on Simultaneous Approximation	594
15.5 Further Results and Conjectures	598
Exercises	606
References	615
Index	629