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**From Group Actions to Determinant Bundles
Using (Heat-kernel) Renormalization Techniques***Sylvie Paycha*

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