

# Table of Contents

<b>Introduction</b> .....	1
<b>1 Differentiable manifolds</b> .....	3
1.1 Embedded manifolds in $\mathbb{R}^N$ .....	4
1.2 The tangent space .....	7
1.3 The derivative of a differentiable function .....	8
1.4 Tangent and cotangent bundles of a manifold .....	9
1.5 Discontinuous action of a group on a manifold .....	9
1.6 Immersions and embeddings. Submanifolds .....	11
1.7 Partition of unity .....	12
<b>2 Vector fields, differential forms and tensor fields</b> .....	13
2.1 Lie derivative of tensor fields .....	16
2.2 The Henri Cartan formula .....	20
<b>3 Pseudo-Riemannian manifolds</b> .....	23
3.1 Affine connections .....	25
3.2 The Levi-Civita connection .....	29
3.3 Tubular neighborhood .....	34
3.4 Curvature .....	36
3.5 E. Cartan structural equations of a connection .....	50
<b>4 Newtonian mechanics</b> .....	55
4.1 Galilean space-time structure and Newton equations .....	55
4.2 Critical remarks on Newtonian mechanics .....	60
<b>5 Mechanical systems on Riemannian manifolds</b> .....	61
5.1 The generalized Newton law .....	61
5.2 The Jacobi Riemannian metric .....	63
5.3 Mechanical systems as second order vector fields .....	66
5.4 Mechanical systems with holonomic constraints .....	68
5.5 Some classical examples .....	70
5.6 The dynamics of rigid bodies .....	78
5.7 Dynamics of pseudo-rigid bodies .....	102
5.8 Dissipative mechanical systems .....	107

<b>6</b>	<b>Mechanical systems with non-holonomic constraints</b> .....	111
6.1	D'Alembert principle .....	111
6.2	Orientability of a distribution and conservation of volume ...	119
6.3	Semi-holonomic constraints .....	123
6.4	The attractor of a dissipative system .....	123
<b>7</b>	<b>Hyperbolicity and Anosov systems. Vakonomic mechanics</b>	127
7.1	Hyperbolic and partially hyperbolic structures .....	127
7.2	Vakonomic mechanics .....	133
7.2.1	Some Hilbert manifolds .....	135
7.2.2	Lagrangian functionals and $\mathcal{D}$ -spaces .....	136
7.3	D'Alembert versus vakonomics .....	136
7.4	Study of the $\mathcal{D}$ -spaces .....	137
7.4.1	The tangent spaces of $H^1(M, \mathcal{D}, [a_0, a_1], m_0)$ .....	137
7.4.2	The $\mathcal{D}$ -space $H^1(M, \mathcal{D}, [a_0, a_1], m_0, m_1)$ . Singular curves	140
7.5	Equations of motion in vakonomic mechanics .....	142
<b>8</b>	<b>Special relativity</b> .....	145
8.1	Lorentz manifolds .....	145
8.2	The quadratic map of $\mathbb{R}_1^{n+1}$ .....	147
8.3	Time-cones and time-orientability of a Lorentz manifold .....	150
8.4	Lorentz geometry notions in special relativity .....	153
8.5	Minkowski space-time geometry .....	155
8.6	Lorentz and Poincaré groups .....	162
<b>9</b>	<b>General relativity</b> .....	165
9.1	Einstein equation .....	165
9.2	Geometric aspects of the Einstein equation .....	166
9.3	Schwarzschild space-time .....	169
9.4	Schwarzschild horizon .....	175
9.5	Light rays, Fermat principle and the deflection of light .....	175
<b>A</b>	<b>Hamiltonian and Lagrangian formalisms</b> .....	183
A.1	Hamiltonian systems .....	183
A.2	Euler-Lagrange equations .....	185
<b>B</b>	<b>Möbius transformations and the Lorentz group</b> .....	195
B.1	The Lorentz group .....	195
B.2	Stereographic projection .....	199
B.3	Complex structure of $S^2$ .....	201
B.4	Möbius transformations .....	204
B.5	Möbius transformations and the proper Lorentz group .....	207
B.6	Lie algebra of the Lorentz group .....	211
B.7	Spinors .....	215
B.8	The sky of a rapidly moving observer .....	217

<b>C</b>	<b>Quasi-Maxwell form of Einstein's equation</b> .....	223
	C.1 Stationary regions, space manifold and global time.....	223
	C.2 Connection forms and equations of motion.....	226
	C.3 Stationary Maxwell equations.....	230
	C.4 Curvature forms and Ricci tensor.....	231
	C.5 Quasi-Maxwell equations.....	235
	C.6 Examples.....	239
<b>D</b>	<b>Viscosity solutions and Aubry–Mather theory</b> .....	245
	D.1 Optimal control and time independent problems.....	245
	D.2 Hamiltonian systems and the Hamilton–Jacobi theory.....	249
	D.3 Aubry–Mather theory.....	251
	<b>References</b> .....	259
	<b>Index</b> .....	263