Table of Contents

Preface vii

Introduction to Global Class Field Theory 1

I. Basic Tools and Notations 7
§1 Places of K 9
§2 Embeddings of a Number Field in its Completions 12
§3 Number and Ideal Groups 21
a) The Local Case: The Group $K_v^\times$ 21
b) The Global Case: Numbers, Ideals, and Units 23
§4 Idèle Groups — Generalized Class Groups 27
a) Idèle Groups — Topology 28
b) Generalized Class Groups — Rank Formulas 37
§5 Reduced Idèles — Topological Aspects 45
a) The Fundamental Exact Sequence 45
b) Topological Lemmas 49
c) Characters of Profinite Groups 53
§6 Kummer Extensions 54
a) Algebraic Kummer Theory 54
b) Arithmetic Aspects of Kummer Theory 59

II. Reciprocity Maps — Existence Theorems 65
§1 The Local Reciprocity Map — Local Class Field Theory 65
a) Decomposition of Places: Local and Global Cases 66
b) Local Class Field Theory Correspondence 74
c) Local Conductors and Norm Groups 80
d) Infinite Local Class Field Theory 86
§2 Idèle Groups in an Extension L/K 91
a) Canonical Injection of C_K in C_L 91
b) Relations Between Local and Global Norms 92
c) Galois Structure of J_L: Semi-Local Theory 94
d) Local Norm Groups — The Non-Galois Case 98
§3 Global Class Field Theory: Idelic Version 104
a) Global Reciprocity Map — The Product Formula — Global Class Field Theory Correspondence 104

b) Global Class Field Theory in $\overline{K}^{\mathrm{ab}}/K$ 121
§4 Global Class Field Theory: Class Group Version 125
a) Global Norm Conductor — Properties 125
b) Artin's Reciprocity Map — Reciprocity Law — Global Computation of Hasse Symbols — Decomposition Law 130
§5 Ray Class Fields — Hilbert Class Fields 143
a) Elementary Properties — Decomposition Law 144
b) Rank Formulas — The Reflection Theorem 152
c) Class Field Theory Over $\mathbb{Q}$ 161
d) Congruence Groups 164
e) Norm Action on Generalized Class Groups 164
f) The Principal Ideal Theorem — Hilbert Towers 168
§6 The Hasse Principle — For Norms — For Powers 176
§7 Symbols Over Number Fields — Hilbert and Regular Kernels 195

III. Abelian Extensions with Restricted Ramification — Abelian Closure 221
§1 Generalities on H_T^S/H^S and its Subextensions 221
a) Description of $\mathrm{Gal}(K_{(\mathfrak{m})}{}^S/H^S)$ 221
b) The Case of p-Extensions 226
c) The Structure of $\mathrm{Gal}(H_T^S/H^S)$ — p-Adic Ranks 233
§2 Computation of $\mathcal{A}_T^S := \mathrm{Gal}(H_{T(p)}^S/K)$ and $\mathcal{T}_T^S := \mathrm{tor}_{\mathbb{Z}_p}(\mathcal{A}_T^S)$. 240
a) $\mathbb{Z}_p$-Free-Extensions — Logarithms 240
b) $\mathcal{A}_T^S$ as an Infinitesimal Ray Class Group 243
c) Computation of $\mathcal{T}_T^S$ 250
d) Class Field Theory Correspondence in $H_T^{\mathrm{res}}{}_{(p)}/K$ 256
§3 Compositum of the S-Split $\mathbb{Z}_p$-Extensions — The p-Adic Conjecture 258
a) p-Adic Ranks: The Leopoldt–Jaulent–Roy Conjecture . 258
b) The Galois Case 264
c) The Monogeneous Case 268
§4 Structure Theorems for the Abelian Closure of K 274
a) Deployment of $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}{}_{(p)}/H_p^{\mathrm{ord}}{}_{(p)})$ 275
b) Triviality Criterion for $\mathcal{T}_T^{\mathrm{ord}}$: When is $\mathcal{G}_T^{\mathrm{ord}}$ Pro-p-Free? 282
c) The Schmidt–Chevalley Theorem — Inertia Groups in $\overline{K}^{\mathrm{ab}}{}_{(p)}/K$ 287
d) Galois Diagram for $\overline{K}^{\mathrm{ab}}{}_{(p)}/K$ — Structure of the Connected Component D_0 — The Fundamental Equality: $\overline{K}_v^{\mathrm{ab}} = (\overline{K}^{\mathrm{ab}})_v$ 291
e) Decomposition Law of Wild Places in $\overline{K}^{\mathrm{ab}}{}_{(p)}/H_p^{\mathrm{ord}}{}_{(p)}$. 300
f) The Strong p-Adic Conjecture — Other p-Adic Aspects 305
g) Structural Properties of $\overline{G}^{\mathrm{ab}}$ — Divisibility of the Connected Component — Cyclic Embedding Criterion 323

h) The Grunwald–Wang Theorem — Weak Deployment Theorem for Decomposition Groups 330
§5 Explicit Computations in Incomplete p-Ramification 342
§6 Initial Radical of the $\mathbb{Z}_p$-Extensions 348
§7 The Logarithmic Class Group 354

IV. Invariant Class Groups in p-Ramification — Genus Theory 361
§1 Reduction to the Case of p-Ramification 362
§2 Injectivity of the Transfer Map $\mathcal{A}_K^{\mathrm{ord}} \longrightarrow \mathcal{A}_L^{\mathrm{ord}}$ 363
§3 Determination of $(\mathcal{A}_L^{\mathrm{ord}})^G$ and $(\mathcal{T}_L^{\mathrm{ord}})^G$ — p-Rational Fields .. 365
a) Invariant Classes Formulas 366
b) p-Primitive Ramification — p-Rationality 371
§4 Genus Theory with Ramification and Decomposition 375
a) Computation of the Number of Genera — Examples .. 377
b) The Genus Exact Sequence 390
c) Central Classes — Knot Groups 398

V. Cyclic Extensions with Prescribed Ramification 407
§1 Study of an Example 408
§2 Construction of a Governing Field 410
a) Solution to the Cyclic Case of Degree p 412
b) Minimal Ramification Sets 421
c) Approach to the Cyclic Case of Degree p^e 423
d) Solution to the Weak Form 432
§3 Conclusion and Perspectives 434

Appendix: Arithmetical Interpretation of $H^2(\mathcal{G}_T^S, \mathbb{Z}/p^e\mathbb{Z})$. 441
§1 A General Approach by Class Field Theory 442
a) Study of $\mathrm{Ker}(H^2(G,\mathbb{Z}/p^e\mathbb{Z}) \xrightarrow{\mathrm{Inf}} H^2(\mathcal{G}_T^S,\mathbb{Z}/p^e\mathbb{Z}))$ 443
b) Study of $H^2(G,\mathbb{Z}/p^e\mathbb{Z})$ — The Schur Multiplier 444
c) A Class Field Theory Formula for $|\mathrm{Inf}(H^2(G.\mathbb{Z}/p^e\mathbb{Z}))|$ 449
§2 Complete p-Ramification Without Finite Decomposition 450
§3 The General Case — Infinitesimal Knot Groups 453
a) Infinitesimal Computations 454
b) Infinitesimal Knot Groups — The Number of Relations — A Generalization of Šafarevič's Results 456
c) Finite Generalized p-Class Fields Towers 460
d) A Lower Bound for $\mathrm{rk}_p(H^2(\mathcal{G}_T^S,\mathbb{Z}/p\mathbb{Z}))$ — Conclusion . 463

Bibliography 467

Index of Notations 481

General Index 487