# Contents


**1 Introduction** ..... 1
1.1 A convex functional ..... 1
1.2 A functional with supercritical growth ..... 2
1.3 Construction of the transformations ..... 5

**2 Uniqueness of critical points (I)** ..... 9
2.1 One-parameter transformation groups ..... 9
2.2 Variational sub-symmetries and uniqueness of critical points .. 10
2.3 Uniqueness results for critical points of constrained functionals 12
2.4 First order variational integrals ..... 12
2.5 Classical uniqueness results ..... 13
2.5.1 Convex functionals ..... 13
2.5.2 Uniqueness of a saddle point ..... 14
2.5.3 Strict variational sub-symmetry w.r.t. an affine subspace 17
2.5.4 Uniqueness of positive solutions for sublinear problems . 21
2.5.5 Simplicity of the first eigenvalue ..... 24

**3 Uniqueness of critical points (II)** ..... 27
3.1 Riemannian manifolds ..... 27
3.2 The total space $M \times \mathbb{R}^k$ ..... 30
3.3 One-parameter transformation groups on $M \times \mathbb{R}^k$ ..... 31
3.4 Action of transformation groups on functions ..... 32
3.5 Rate of change of derivatives and volume-forms ..... 35
3.6 Rate of change of first-order variational functionals ..... 39
3.6.1 Partial derivatives of Lagrangians ..... 39
3.6.2 The rate of change formula ..... 41
3.6.3 Noether's formula and Pohožaev's identity ..... 43
3.7 Admissible transformation groups ..... 44
3.8 Rate of change formula for solutions ..... 46
3.9 Variational sub-symmetries ..... 48
3.10 Uniqueness of critical points ..... 50

3.11 Uniqueness of critical points for constrained functionals ..... 53
3.11.1 Functional constraints ..... 53
3.11.2 Pointwise constraints ..... 54
3.12 Differentiability of the group orbits ..... 56

**4 Variational problems on Riemannian manifolds** ..... 59
4.1 Example manifolds and their representations ..... 59
4.2 Supercritical boundary value problems ..... 61
4.2.1 A weak substitute for the vector-field $\boldsymbol{x}$ ..... 62
4.2.2 Critical points of a free functional ..... 63
4.2.3 Critical points of constrained functionals ..... 66
4.2.4 Applications ..... 68
4.3 Harmonic maps ..... 71
4.4 Supercritical boundary value problems: revisited ..... 73
4.4.1 A better substitute for the vector-field $\boldsymbol{x}$? ..... 74
4.4.2 Conformal vector-fields and conformal maps ..... 76
4.4.3 Yamabe's equation ..... 79
4.4.4 Yamabe's equation with boundary terms ..... 80
4.4.5 Conformal vector fields on conformally flat manifolds .. 80
4.4.6 The bifurcation problem on $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ ..... 83
4.4.7 The bifurcation problem on rotation surfaces ..... 84
4.5 Harmonic maps into conformally flat manifolds ..... 86

**5 Scalar problems in Euclidean space** ..... 89
5.1 Extensions of Pohožaev's result to more general domains ..... 89
5.1.1 Nonlinear Neumann boundary conditions ..... 94
5.1.2 Extension to operators of $q$-Laplacian type ..... 97
5.1.3 Extension to the mean-curvature operator ..... 99
5.2 Uniqueness of non-zero solutions ..... 100
5.3 The subcritical case ..... 106
5.4 Perturbations of conformally contractible domains ..... 107
5.5 Uniqueness in the presence of radial symmetry ..... 110
5.5.1 Radially symmetric problems on $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$ ..... 112
5.5.2 The radially symmetric $q$-Laplacian ..... 119
5.5.3 Partial radial symmetry ..... 121
5.6 Notes on further results ..... 123

**6 Vector problems in Euclidean space** ..... 127
6.1 The Emden-Fowler system ..... 127
6.2 Boundary displacement problem in nonlinear elasticity ..... 130
6.2.1 Uniqueness for the boundary displacement problem (compressible case) ..... 132
6.2.2 Uniqueness for the boundary displacement problem (incompressible case) ..... 134
6.3 A uniqueness result in dimension two ..... 134

6.4 H. Wente's uniqueness result for closed surfaces of prescribed mean curvature ..... 137

**A Fréchet-differentiability** ..... 139

**B Lipschitz-properties of $g_\epsilon$ and $\Omega_\epsilon$** ..... 141

**References** ..... 145

**Index** ..... 151