Table of Contents

Introduction 1

I The Hahn-Banach-Lagrange theorem and some consequences

1 The Hahn–Banach–Lagrange theorem 15
2 Applications to functional analysis 23
3 A minimax theorem 24
4 The dual and bidual of a normed space 25
5 Excess, duality gap, and minimax criteria for weak compactness 28
6 Sharp Lagrange multiplier and KKT results 32

II Fenchel duality

7 A sharp version of the Fenchel Duality theorem 41
8 Fenchel duality with respect to a bilinear form — locally convex spaces 44
9 Some properties of $\frac{1}{2}\|\cdot\|^2$ 49
10 The conjugate of a sum in the locally convex case 51
11 Fenchel duality vs the conjugate of a sum 54
12 The restricted biconjugate and Fenchel–Moreau points ... 58
13 Surrounding sets and the dom lemma 60
14 The $\ominus$–theorem 62
15 The Attouch–Brezis theorem 65
16 A bivariate Attouch–Brezis theorem 67

III Multifunctions, SSD spaces, monotonicity and Fitzpatrick functions

17 Multifunctions, monotonicity and maximality 71

18 Subdifferentials are maximally monotone 74

19 SSD spaces, q–positive sets and BC–functions 79

20 Maximally q–positive sets in SSD spaces 86

21 SSDB spaces 88

22 The SSD space $E \times E^*$ 93

23 Fitzpatrick functions and fitzpatrifications 99

24 The maximal monotonicity of a sum 103

IV Monotone multifunctions on general Banach spaces

25 Monotone multifunctions with bounded range 107

26 A general local boundedness theorem 108

27 The six set theorem and the nine set theorem 108

28 $D(S_\varphi)$ and various hulls 111

V Monotone multifunctions on reflexive Banach spaces

29 Criteria for maximality, and Rockafellar's surjectivity theorem 117

30 Surjectivity and an abstract Hammerstein theorem 123

31 The Brezis–Haraux condition 125

32 Bootstrapping the sum theorem 128

33 The $>$ six set and the $>$ nine set theorems for pairs of multifunctions 130

34 The Brezis–Crandall–Pazy condition 132

VI Special maximally monotone multifunctions

35 The norm–dual of the space $E \times E^*$ and $\widetilde{\mathrm{BC}}$–functions ... 139
36 Subclasses of the maximally monotone multifunctions ... 147
37 First application of Theorem 35.8: type (D) implies type (FP) ... 153
38 $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, $\mathcal{T}_{\mathcal{CLBN}}(B^*)$ and type (ED) ... 154
39 Second application of Theorem 35.8: type (ED) implies type (FPV) ... 157
40 Final applications of Theorem 35.8: type (ED) implies strong ... 158
41 Strong maximality and coercivity ... 159
42 Type (ED) implies type (ANA) and type (BR) ... 161
43 The closure of the range ... 167
44 The sum problem and the closure of the domain ... 170
45 The biconjugate of a maximum and $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ ... 172
46 Maximally monotone multifunctions with convex graph ... 180
47 Possibly discontinuous positive linear operators ... 183
48 Subtler properties of subdifferentials ... 188
49 Saddle functions and type (ED) ... 192

VII The sum problem for general Banach spaces

50 Introductory comments ... 197
51 Voisei's theorem ... 197
52 Sums with normality maps ... 198
53 A theorem of Verona–Verona ... 199

VIII Open problems 203

IX Glossary of classes of multifunctions 205

X A selection of results 207

References 233

Subject index 239

Symbol index 243