

---

# Contents

<b>Introduction</b> .....	1
<b>1 Generalities</b> .....	9
1.1 Setup .....	10
1.2 Valuations .....	10
1.3 Krull Valuations .....	11
1.4 Plane Curves .....	13
1.5 Examples of Valuations .....	13
1.5.1 The Multiplicity Valuation .....	14
1.5.2 Monomial Valuations .....	14
1.5.3 Divisorial Valuations .....	15
1.5.4 Quasimonomial Valuations .....	16
1.5.5 Curve Valuations .....	17
1.5.6 Exceptional Curve Valuations .....	18
1.5.7 Infinitely Singular Valuations .....	18
1.6 Valuations Versus Krull Valuations .....	19
1.7 Sequences of Blowups and Krull Valuations .....	21
<b>2 MacLane's Method</b> .....	25
2.1 Sequences of Key Polynomials .....	26
2.1.1 Key Polynomials .....	26
2.1.2 From SKP's to Valuations I .....	28
2.1.3 Proof of Theorem 2.8 .....	29
2.1.4 From SKP's to Valuations II .....	33
2.2 Classification .....	34
2.3 Graded Rings and Numerical Invariants .....	35
2.3.1 Homogeneous Decomposition I .....	35
2.3.2 Homogeneous Decomposition II .....	37
2.3.3 Value Semigroups and Numerical Invariants .....	38
2.4 From Valuations to SKP's .....	39
2.5 A Computation .....	40

<b>3</b>	<b>Tree Structures</b> .....	43
3.1	Trees .....	44
3.1.1	Rooted Nonmetric Trees .....	44
3.1.2	Nonmetric Trees .....	46
3.1.3	Parameterized Trees .....	47
3.1.4	The Weak Topology .....	47
3.1.5	Metric Trees .....	49
3.1.6	Trees from Ultrametric Spaces .....	51
3.1.7	Trees from Simplicial Trees .....	51
3.1.8	Trees from $\mathbf{Q}$ -trees .....	52
3.2	Nonmetric Tree Structure on $\mathcal{V}$ .....	54
3.2.1	Partial Ordering .....	54
3.2.2	Dendrology .....	56
3.2.3	A Model Tree for $\mathcal{V}$ .....	58
3.3	Parameterization of $\mathcal{V}$ by Skewness .....	58
3.3.1	Skewness .....	59
3.3.2	Parameterization .....	59
3.3.3	Proofs .....	61
3.3.4	Tree Metrics .....	62
3.4	Multiplicities .....	63
3.5	Approximating Sequences .....	66
3.6	Thinness .....	67
3.7	Value Semigroups and Approximating Sequences .....	69
3.8	Balls of Curves .....	72
3.8.1	Valuations Through Intersections .....	72
3.8.2	Balls of Curves .....	73
3.9	The Relative Tree Structure .....	74
3.9.1	The Relative Valuative Tree .....	75
3.9.2	Relative Parameterizations .....	77
3.9.3	Balls of Curves .....	79
3.9.4	Homogeneity .....	80
<b>4</b>	<b>Valuations Through Puiseux Series</b> .....	81
4.1	Puiseux Series and Valuations .....	81
4.2	Tree Structure .....	84
4.2.1	Nonmetric Tree Structure .....	84
4.2.2	Puiseux Parameterization .....	86
4.2.3	Multiplicities .....	87
4.3	Galois Action .....	88
4.3.1	The Galois Group .....	88
4.3.2	Action on $\widehat{\mathcal{V}}_x$ .....	88
4.3.3	The Orbit Tree .....	89
4.4	A Tale of Two Trees .....	89
4.4.1	Minimal Polynomials .....	90
4.4.2	The Morphism .....	90
4.4.3	Proof .....	91

4.5	The Berkovich Projective Line .....	94
4.6	The Bruhat-Tits Metric .....	95
4.7	Dictionary .....	95
<b>5</b>	<b>Topologies</b> .....	<b>97</b>
5.1	The Weak Topology .....	98
5.1.1	The Equivalence .....	98
5.1.2	Properties .....	98
5.2	The Strong Topology on $\mathcal{V}$ .....	99
5.2.1	Strong Topology I .....	99
5.2.2	Strong Topology II .....	100
5.2.3	The Equivalence .....	100
5.2.4	Properties .....	101
5.3	The Strong Topology on $\mathcal{V}_{\text{qm}}$ .....	102
5.4	Thin Topologies .....	103
5.5	The Zariski Topology .....	104
5.5.1	Definition .....	104
5.5.2	Recovering $\mathcal{V}$ from $\mathcal{V}_K$ .....	105
5.6	The Hausdorff-Zariski Topology .....	106
5.6.1	Definition .....	106
5.6.2	The $\overline{\mathbf{N}}$ -tree Structure on $\mathcal{V}_K$ .....	107
5.7	Comparison of Topologies .....	108
5.7.1	Topologies .....	108
5.7.2	Metrics .....	109
<b>6</b>	<b>The Universal Dual Graph</b> .....	<b>111</b>
6.1	Nonmetric Tree Structure .....	112
6.1.1	Compositions of Blowups .....	112
6.1.2	Dual Graphs .....	112
6.1.3	The $\mathbf{Q}$ -tree .....	113
6.1.4	Tangent Spaces .....	115
6.1.5	The $\mathbf{R}$ -tree .....	116
6.2	Infinitely Near Points .....	116
6.2.1	Definitions and Main Results .....	116
6.2.2	Proofs .....	118
6.3	Parameterization and Multiplicity .....	121
6.3.1	Farey Weights and Parameters .....	122
6.3.2	Multiplicities .....	123
6.4	The Isomorphism .....	125
6.5	Proof of the Isomorphism .....	126
6.5.1	Step 1: $\Phi : \Gamma^* \rightarrow \mathcal{V}_{\text{div}}$ is bijective .....	126
6.5.2	Step 2: $A \circ \Phi = A$ .....	127
6.5.3	Step 3: $\Phi$ and $\Phi^{-1}$ Are Order Preserving .....	129
6.5.4	Step 4: $\Phi$ Preserves Multiplicity .....	131

6.6	Applications	132
6.6.1	Curvettes	132
6.6.2	Centers of Valuations and Partitions of $\mathcal{V}$	133
6.6.3	Potpourri on Divisorial Valuations	135
6.6.4	Monomialization	139
6.7	The Dual Graph of the Minimal Desingularization	140
6.7.1	The Embedding of $\Gamma_C^*$ in $\Gamma^*$	141
6.7.2	Construction of $\Gamma_C$ from the Equisingularity Type of $C$	142
6.8	The Relative Tree Structure	145
6.8.1	The Relative Dual Graph	145
6.8.2	Weights, Parameterization and Multiplicities	146
6.8.3	The Isomorphism	147
6.8.4	The Contraction Map at a Free Point	148
<b>7</b>	<b>Tree Measures</b>	<b>151</b>
7.1	Outline	151
7.1.1	The Unbranched Case	152
7.1.2	The General Case	152
7.1.3	Organization	154
7.2	More on the Weak Topology	155
7.2.1	Definition	156
7.2.2	Basic properties	156
7.2.3	Subtrees	157
7.2.4	Connectedness	158
7.2.5	Compactness	160
7.3	Borel Measures	161
7.3.1	Basic Properties	161
7.3.2	Radon Measures	161
7.3.3	Spaces of Measures	162
7.3.4	The Support of a Measure	163
7.3.5	A Generating Algebra	163
7.3.6	Every Complex Borel Measure is Radon	164
7.4	Functions of Bounded Variation	165
7.4.1	Definitions	165
7.4.2	Decomposition	166
7.4.3	Limits and Continuity	168
7.4.4	The Space $\mathcal{N}$	169
7.4.5	Finite Trees	170
7.4.6	Proofs	171
7.5	Representation Theorem I	173
7.5.1	First Step	173
7.5.2	Second Step: from Functions to Measures	174
7.5.3	Total Variation	175
7.6	Complex Tree Potentials	176
7.6.1	Definition	176
7.6.2	Directional Derivatives	177

7.7	Representation Theorem II	178
7.8	Atomic Measures	179
7.9	Positive Tree Potentials	180
7.9.1	Definition	180
7.9.2	Jordan Decompositions	182
7.10	Weak Topologies and Compactness	183
7.11	Restrictions to Subtrees	185
7.12	Inner Products	186
7.12.1	Hausdorff Measure	186
7.12.2	The Positive Case	187
7.12.3	Properties	189
7.12.4	The Complex Case	190
7.12.5	Topologies and Completeness	192
<b>8</b>	<b>Applications of the Tree Analysis</b>	<b>193</b>
8.1	Zariski's Theory of Complete Ideals	194
8.1.1	Basic Properties	194
8.1.2	Normalized Blowup	197
8.1.3	Integral Closures	198
8.1.4	Multiplicities	199
8.2	The Voûte Étoilée	200
8.2.1	Definition	200
8.2.2	Cohomology	201
8.2.3	Intersection Product	202
8.2.4	Associated Complex Tree Potentials	203
8.2.5	Isometric Embedding	205
8.2.6	Cohomology Groups	207
	<b>Appendix</b>	<b>211</b>
A	Infinitely Singular Valuations	211
A.1	Characterizations	212
A.2	Constructions	213
B	The Tangent Space at a Divisorial Valuation	214
C	Classification	217
D	Combinatorics of Plane Curve Singularities	218
D.1	Zariski's Terminology for Plane Curve Singularities	219
D.2	The Eggers Tree	221
E	What are the Essential Assumptions on the Ring $R$ ?	224
E.1	Completeness	224
E.2	The Residue Field	225
	<b>References</b>	<b>227</b>
	<b>Index</b>	<b>231</b>