

# Contents

<b>Preface</b>	<b>xi</b>
<b>Note to the Reader</b>	<b>xx</b>
<b>Terminology</b>	<b>xxi</b>
<b>I Valuation Theory</b>	<b>1</b>
1 Marot Rings . . . . .	1
1.1 Marot Rings . . . . .	2
1.2 Large Quotient Ring . . . . .	3
1.3 Rings With Large Jacobson Radical . . . . .	4
2 Manis Valuation Rings . . . . .	5
2.1 Manis Valuation Rings . . . . .	5
2.2 Manis Valuations . . . . .	9
2.3 The Approximation Theorem For Discrete Manis Valuations	14
3 Valuation Rings and Valuations . . . . .	17
3.1 Valuation Rings . . . . .	17
3.2 Subrings and Overrings of Valuation Rings . . . . .	18
3.3 Valuations . . . . .	20
3.4 Composite Valuations . . . . .	23
3.5 Discrete Valuations . . . . .	24
3.6 Existence of Valuations of the Second Kind . . . . .	26
4 The Approximation Theorem For Independent Valuations . . . . .	27
5 Extensions of Valuations . . . . .	29
5.1 Existence of Extensions . . . . .	29
5.2 Reduced Ramification Index and Residue Degree . . . . .	30
5.3 Extension of Composite Valuations . . . . .	30
6 Extending Valuations to Algebraic Overfields . . . . .	31
6.1 Some General Results . . . . .	31
6.2 The Formula $ef \leq n$ . . . . .	33
6.3 The Formula $\sum e_i f_i \leq n$ . . . . .	36
6.4 The Formula $\sum e_i f_i = n$ . . . . .	38
7 Extensions of Discrete Valuations . . . . .	40

	7.1	Intersections of Discrete Valuation Rings . . . . .	40
	7.2	Extensions of Discrete Valuations . . . . .	40
	7.3	Some Classes of Extensions . . . . .	41
	7.4	Quadratic Number Fields . . . . .	44
8		Ramification Theory of Valuations . . . . .	46
	8.1	Generalities . . . . .	46
	8.2	The Value Groups $\Gamma$ , $\Gamma_{\mathbb{Z}}$ and $\Gamma_{\mathbb{T}}$ . . . . .	47
	8.3	The Ramification Group . . . . .	49
9		Extending Valuations to Non-Algebraic Overfields . . . . .	53
10		Valuations of Algebraic Function Fields . . . . .	56
11		Valuations Dominating a Local Domain . . . . .	59
<b>II</b>		<b>One-Dimensional Semilocal Cohen-Macaulay Rings</b>	<b>67</b>
1		Transversal Elements . . . . .	67
	1.1	Adic topologies . . . . .	67
	1.2	The Hilbert Polynomial . . . . .	69
	1.3	Transversal Elements . . . . .	70
2		Integral Closure of One-Dimensional Semilocal Cohen-Macaulay Rings . . . . .	74
	2.1	Invertible Modules . . . . .	74
	2.2	The Integral Closure . . . . .	75
	2.3	Integral Closure and Manis Valuation Rings . . . . .	77
3		One-Dimensional Analytically Unramified and Analytically Irre- ducible CM-Rings . . . . .	79
	3.1	Two Length Formulae . . . . .	79
	3.2	Divisible Modules . . . . .	80
	3.3	Compatible Extensions . . . . .	81
	3.4	Criteria for One-Dimensional Analytically Unramified and Analytically Irreducible CM-Rings . . . . .	84
4		Blowing up Ideals . . . . .	88
	4.1	The Blow-up Ring $R^a$ . . . . .	88
	4.2	Integral Closure . . . . .	91
	4.3	Stable Ideals . . . . .	92
5		Infinitely Near Rings . . . . .	96
<b>III</b>		<b>Differential Modules and Ramification</b>	<b>101</b>
1		Introduction . . . . .	101
2		Norms and Traces . . . . .	104
	2.1	Some Linear Algebra . . . . .	105
	2.2	Determinant and Characteristic Polynomial . . . . .	106
	2.3	The Trace Form . . . . .	109
3		Formally Unramified and Unramified Extensions . . . . .	113
	3.1	The Branch Locus . . . . .	113
	3.2	Some Ramification Criteria . . . . .	115

	3.3	Ramification for Local Rings and Applications . . . . .	120
	3.4	Discrete Valuation Rings and Ramification . . . . .	124
4		Unramified Extensions and Discriminants . . . . .	125
5		Ramification For Quasilocal Rings . . . . .	131
6		Integral Closure and Completion . . . . .	134
<b>IV</b>		<b>Formal and Convergent Power Series Rings</b>	<b>143</b>
1		Formal Power Series Rings . . . . .	143
2		Convergent Power Series Rings . . . . .	146
3		Weierstraß Preparation Theorem . . . . .	147
	3.1	Weierstraß Division Theorem . . . . .	147
	3.2	Weierstraß Preparation Theorem and Applications . . . . .	152
4		The Category of Formal and Analytic Algebras . . . . .	157
	4.1	Local $k$ -algebras . . . . .	157
	4.2	Morphisms of Formal and Analytic Algebras . . . . .	157
	4.3	Integral Extensions . . . . .	162
	4.4	Noether Normalization . . . . .	163
5		Extensions of Formal and Analytic Algebras . . . . .	166
<b>V</b>		<b>Quasiordinary Singularities</b>	<b>169</b>
1		Fractionary Power Series . . . . .	169
	1.1	Generalities . . . . .	169
	1.2	Intermediate Fields . . . . .	171
	1.3	Intermediate Fields Generated by a Fractionary Power Series	175
2		The Jung-Abhyankar Theorem: Formal Case . . . . .	177
3		The Jung-Abhyankar Theorem: Analytic Case . . . . .	181
4		Quasiordinary Power Series . . . . .	182
5		A Generalized Newton Algorithm . . . . .	190
	5.1	The Algorithm . . . . .	190
	5.2	An Example . . . . .	195
6		Strictly Generated Semigroups . . . . .	198
	6.1	Generalities . . . . .	198
	6.2	Strictly Generated Semigroups . . . . .	202
<b>VI</b>		<b>The Singularity <math>Z^q = XY^p</math></b>	<b>205</b>
1		Hirzebruch-Jung Singularities . . . . .	205
2		Semigroups and Semigroup Rings . . . . .	210
	2.1	Generalities . . . . .	210
	2.2	Integral Closure of Semigroup Rings . . . . .	213
3		Continued Fractions . . . . .	215
	3.1	Continued Fractions . . . . .	216
	3.2	Hirzebruch-Jung Continued Fractions . . . . .	218
4		Two-Dimensional Cones . . . . .	222
	4.1	Two-dimensional Cones and Semigroups . . . . .	222
	4.2	The Boundary Polygon of $\sigma$ and the Ideal of $X_\sigma$ . . . . .	225

5	Resolution of Singularities . . . . .	235
5.1	Some Useful Formulae . . . . .	235
5.2	The Case $p = 1$ . . . . .	237
5.3	The General Case . . . . .	238
5.4	Counting Singularities of the Blow-up . . . . .	244
<b>VII</b>	<b>Two-Dimensional Regular Local Rings</b>	<b>247</b>
1	Ideal Transform . . . . .	247
1.1	Generalities . . . . .	247
1.2	Ideal Transforms . . . . .	249
2	Quadratic Transforms and Ideal Transforms . . . . .	252
2.1	Generalities . . . . .	252
2.2	Quadratic Transforms and the First Neighborhood . . . . .	255
2.3	Ideal Transforms . . . . .	257
2.4	Valuations Dominating $R$ . . . . .	260
3	Complete Ideals . . . . .	261
3.1	Generalities . . . . .	261
3.2	Complete Ideals as Intersections . . . . .	263
3.3	When Does $\mathfrak{m}$ Divide a Complete Ideal? . . . . .	268
3.4	An Existence Theorem . . . . .	271
4	Factorization of Complete Ideals . . . . .	273
4.1	Preliminary Results . . . . .	273
4.2	Contracted Ideals . . . . .	278
4.3	Unique Factorization . . . . .	281
5	The Predecessors of a Simple Ideal . . . . .	282
6	The Quadratic Sequence . . . . .	287
7	Proximity . . . . .	292
8	Resolution of Embedded Curves . . . . .	296
<b>VIII</b>	<b>Resolution of Singularities</b>	<b>303</b>
1	Blowing up Curve Singularities . . . . .	303
2	Resolution of Surface Singularities I: Jung's Method . . . . .	310
3	Quadratic Dilatations . . . . .	313
3.1	Quadratic Dilatations . . . . .	313
3.2	Quadratic Dilatations and Algebraic Varieties . . . . .	314
4	Quadratic Dilatations of Two-Dimensional Regular Local Rings . . . . .	316
5	Valuations of Algebraic Function Fields in Two Variables . . . . .	320
6	Uniformization . . . . .	324
6.1	Classification of Valuations and Local Uniformization . . . . .	324
6.2	Existence of Subrings Lying Under a Local Ring . . . . .	328
6.3	Uniformization . . . . .	330
7	Resolution of Surface Singularities II: Blowing up and Normalizing . . . . .	334
7.1	Principalization . . . . .	334
7.2	Tangential Ideals . . . . .	339

7.3	The Main Result . . . . .	343
<b>Appendices</b>		<b>345</b>
<b>A</b>	<b>Results from Classical Algebraic Geometry</b>	<b>345</b>
1	Generalities . . . . .	345
1.1	Ideals and Varieties . . . . .	345
1.2	Rational Functions and Maps . . . . .	347
1.3	Coordinate Ring and Local Rings . . . . .	348
1.4	Dominant Morphisms and Closed Embeddings . . . . .	349
1.5	Elementary Open Sets . . . . .	350
1.6	Varieties as Topological Spaces . . . . .	351
1.7	Local Ring on a Subvariety . . . . .	353
2	Affine and Finite Morphisms . . . . .	355
3	Products . . . . .	357
4	Proper Morphisms . . . . .	361
4.1	Space of Irreducible Closed Subsets . . . . .	362
4.2	Varieties and the Functor $t$ . . . . .	365
4.3	Proper Morphisms . . . . .	367
5	Algebraic Cones and Projective Varieties . . . . .	370
6	Regular and Singular Points . . . . .	373
7	Normalization of a Variety . . . . .	378
8	Desingularization of a Variety . . . . .	384
9	Dimension of Fibres . . . . .	385
10	Quasifinite Morphisms and Ramification . . . . .	387
10.1	Quasifinite Morphisms . . . . .	387
10.2	Ramification . . . . .	389
11	Divisors . . . . .	392
12	Some Results on Projections . . . . .	395
13	Blowing up . . . . .	398
14	Blowing up: The Local Rings . . . . .	403
<b>B</b>	<b>Miscellaneous Results</b>	<b>409</b>
1	Ordered Abelian Groups . . . . .	409
1.1	Isolated Subgroups . . . . .	409
1.2	Initial Index . . . . .	411
1.3	Archimedean Ordered Groups . . . . .	412
1.4	The Rational Rank of an Abelian Group . . . . .	415
2	Localization . . . . .	419
3	Integral Extensions . . . . .	421
4	Some Results on Graded Rings and Modules . . . . .	423
4.1	Generalities . . . . .	423
4.2	$M$ -Graded Rings and $M$ -Graded Modules . . . . .	424
4.3	Homogeneous Localization . . . . .	426
4.4	Integral Closure of Graded Rings . . . . .	430

5	Properties of the Rees Ring . . . . .	433
6	Integral Closure of Ideals . . . . .	437
6.1	Generalities . . . . .	437
6.2	Integral Closure of Ideals . . . . .	437
6.3	Integral Closure of Ideals and Valuation Theory . . . . .	440
7	Decomposition Group and Inertia Group . . . . .	442
8	Decomposable Rings . . . . .	448
9	The Dimension Formula . . . . .	449
10	Miscellaneous Results . . . . .	452
10.1	The Chinese Remainder Theorem . . . . .	452
10.2	Separable Noether Normalization . . . . .	453
10.3	The Segre Ideal . . . . .	454
10.4	Adjoining an Indeterminate . . . . .	456
10.5	Divisor Group and Class Group . . . . .	457
10.6	Calculating a Multiplicity . . . . .	457
10.7	A Length Formula . . . . .	458
10.8	Quasifinite Modules . . . . .	459
10.9	Maximal Primary Ideals . . . . .	460
10.10	Primary Decomposition in Non-Noetherian Rings . . . . .	461
10.11	Discriminant of a Polynomial . . . . .	462
	<b>Bibliography</b>	<b>463</b>
	<b>Index of Symbols</b>	<b>475</b>
	<b>Index</b>	<b>478</b>