Contents

Preface ix

Introduction to Symmetric Spaces and Their Compactifications
Lizhen Ji 1
1 Introduction 1
2 The Poincaré disc D 2
3 The bidisc 24
4 Symmetric spaces 32
References 66

Compactifications of Symmetric and Locally Symmetric Spaces
Armand Borel and Lizhen Ji 69
1 Introduction 69
2 Summary 70
3 Geometry of symmetric spaces 74
4 Satake and Furstenberg compactifications 79
5 Alternative constructions of $\overline{X}^{SF}_{\max}$ 86
6 Martin compactifications and Karpelevic compactification 92
7 Geometry of locally symmetric spaces 99
8 Compactifications of locally symmetric spaces 102
9 Satake compactifications of $\Gamma\backslash X$ 109
10 Compactifications in which all open orbits are isomorphic 113
11 Real forms and real points of complex orbits 118
12 The wonderful compactification of G_c/K_c and its real points 121
13 The Oshima–Sekiguchi compactification of G/K 125
14 The wonderful compactification of G_c/H_c and its real points 128
15 Appendix: Galois cohomology and real forms 133
References 135

Restrictions of Unitary Representations of Real Reductive Groups
T. Kobayashi 139
1 Reductive Lie groups 139
1.1 Smallest objects 139

1.2 General linear group $GL(N, \mathbb{R})$ ... 140
1.3 Cartan decomposition ... 140
1.4 Reductive Lie groups ... 141
1.5 Examples of reductive Lie groups ... 142
1.6 Inclusions of groups and restrictions of representations ... 144
2 Unitary representations and admissible representations ... 145
2.1 Continuous representations ... 145
2.2 Example ... 145
2.3 Unitary representations ... 145
2.4 Admissible restrictions ... 146
2.4.1 (Unitarily) discretely decomposable representations ... 146
2.4.2 Admissible representations ... 146
2.4.3 Gelfand–Piateski-Shapiro's theorem ... 146
2.4.4 Admissible restrictions ... 147
2.4.5 Chain rule of admissible restrictions ... 147
2.4.6 Harish-Chandra's admissibility theorem ... 147
2.4.7 Further readings ... 148
3 $SL(2, \mathbb{R})$ and Branching Laws ... 148
3.1 Branching Problems ... 148
3.1.1 Direct integral of Hilbert spaces ... 148
3.1.2 Irreducible decomposition ... 149
3.1.3 Examples ... 150
3.1.4 Branching problems ... 150
3.2 Unitary dual of $SL(2, \mathbb{R})$ ... 150
3.2.1 $SL(2, \mathbb{R})$-action on $\mathbb{P}^1\mathbb{C}$ ... 150
3.2.2 Unitary principal series representations ... 151
3.2.3 Holomorphic discrete series representations ... 151
3.2.4 Restriction and proof for irreducibility ... 152
3.3 Branching laws of $SL(2, \mathbb{R})$ ... 153
3.3.1 Subgroups of $SL(2, \mathbb{R})$ ... 153
3.3.2 Branching laws $G \downarrow K, A, N$ ... 153
3.3.3 Restriction of holomorphic discrete series ... 154
3.3.4 Irreducibility of π_n^+ $(n = 2, 3, 4, \dots)$ ... 155
3.4 $\otimes$-product representations of $SL(2, \mathbb{R})$ ... 155
3.4.1 Tensor product representations ... 155
3.4.2 $\pi_\lambda \otimes \pi_{\lambda'}$ (principal series) ... 156
3.4.3 $\pi_m^+ \otimes \pi_n^+$ (holomorphic discrete series) ... 157
4 $(\mathfrak{g}, K)$-modules and infinitesimal discrete decomposition ... 158
4.1 Category of $(\mathfrak{g}, K)$-modules ... 158
4.1.1 K-finite vectors ... 158
4.1.2 Underlying $(\mathfrak{g}, K)$-modules ... 158
4.1.3 $(\mathfrak{g}, K)$-modules ... 159
4.1.4 Infinitesimally unitary representations ... 159
4.1.5 Scope ... 160
4.1.6 For further reading ... 160

4.2 Infinitesimal discrete decomposition ... 160
4.2.1 Wiener subspace ... 160
4.2.2 Discretely decomposable modules ... 161
4.2.3 Infinitesimally discretely decomposable representations ... 161
4.2.4 Examples ... 162
4.2.5 Unitary case ... 162
4.2.6 Infinitesimal discrete decomposability $\Rightarrow$ discrete decomposability ... 163
4.2.7 K'-admissibility $\Rightarrow$ infinitesimally discrete decomposability 163
4.2.8 For further reading ... 163
5 Algebraic theory of discretely decomposable restrictions ... 163
5.1 Associated varieties ... 164
5.1.1 Graded modules ... 164
5.1.2 Associated varieties of $\mathfrak{g}$-modules ... 164
5.1.3 Associated varieties of G-representations ... 165
5.1.4 Nilpotent cone ... 166
5.2 Restrictions and associated varieties ... 166
5.2.1 Associated varieties of irreducible summands ... 167
5.2.2 G': compact case ... 168
5.2.3 Criterion for infinitesimally discretely decomposable restrictions ... 168
5.3 Examples ... 168
5.3.1 Strategy ... 169
5.3.2 $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{pc}$ ... 169
5.3.3 Interpretations from representation theory ... 171
5.3.4 Case $(G, G_1) = (U(2,2), Sp(1,1))$ (essentially, $(SO(4,2), SO(4,1))$) ... 172
5.3.5 Case $(G, G_2) = (U(2,2), U(2,1) \times U(1))$ ... 173
5.3.6 Case $(G, G_3) = (U(2,2), U(2) \times U(2))$ ... 174
6 Admissible restriction and microlocal analysis ... 174
6.1 Hyperfunction characters ... 174
6.1.1 Finite group ... 175
6.1.2 Dirac's delta function ... 175
6.1.3 Schwartz's distributions ... 175
6.1.4 Sato's hyperfunctions ... 176
6.1.5 Distributions or hyperfunctions ... 176
6.1.6 Strategy ... 177
6.2 Asymptotic K-support ... 178
6.2.1 Asymptotic cone ... 178
6.2.2 Cartan–Weyl highest weight theory ... 179
6.2.3 Asymptotic K-support ... 179
6.2.4 Examples from $SL(2, \mathbb{R})$ ... 180
6.3 Criterion for admissible restriction ... 180
6.3.1 The closed cone $C_K(K')$ ... 180
6.3.2 Symmetric pair ... 180

6.3.3 Criterion for K'-admissibility 181
6.3.4 Sufficient condition for G'-admissibility 181
6.3.5 Remark 182
6.4 Application of symplectic geometry 182
6.4.1 Hamiltonian action 182
6.4.2 Affine varieties 183
6.4.3 Cotangent bundle 183
7 Discretely decomposable restriction of $A_{\mathfrak{q}}(\lambda)$ 184
7.1 Elliptic orbits and geometric quantization 184
7.1.1 Elliptic orbits 184
7.1.2 Complex structure on an elliptic orbit 185
7.1.3 Elliptic coadjoint orbit 186
7.1.4 Geometric quantization à la Schmid–Wong 187
7.1.5 Borel–Weil–Bott theorem 188
7.1.6 Discrete series representations 188
7.1.7 Harish-Chandra's discrete series representations 189
7.1.8 Discrete series representations for symmetric spaces 189
7.2 Restriction of $\Pi(\lambda)$ attached to elliptic orbits 190
7.2.1 Asymptotic K-support, associated variety 190
7.2.2 Restriction to a symmetric subgroup 190
7.3 $U(2,2) \downarrow Sp(1,1)$ 191
7.3.1 Non-holomorphic discrete series representations for $U(2,2)$ 191
7.3.2 Criterion for K'-admissibility for $U(2,2) \downarrow Sp(1,1)$ 192
7.3.3 Associated variety of $\Pi(\lambda)$ 192
8 Applications of branching problems 194
8.1 Understanding representations via restrictions 194
8.1.1 Analysis and Synthesis 194
8.1.2 Cartan–Weyl highest weight theory, revisited 194
8.1.3 Vogan's minimal K-type theory 195
8.1.4 Restrictions to noncompact groups 195
8.2 Construction of representations of subgroups 196
8.2.1 Finite-dimensional representations 196
8.2.2 Highest weight modules 196
8.2.3 Small representations 197
8.3 Branching problems 198
8.4 Global analysis 198
8.4.1 Global analysis and restriction of representations 198
8.4.2 Discrete series and admissible representations 199
8.5 Discrete groups and restriction of unitary representations ... 200
8.5.1 Matsushima–Murakami's formula 200
8.5.2 Vanishing theorem for modular varieties 201
8.5.3 Clifford–Klein problem 202
References 202