

Contents

Preface	xvii
Foreword	xix
Glossary of Notation	xxiii
I First Concepts	1
1 Two Definitions of Lattices	1
1.1 Orders	1
1.2 Equivalence relations and preorderings	2
1.3 Basic order concepts	4
1.4 Ordering and covers	5
1.5 Order diagrams	6
1.6 Order constructions	7
1.7 Two more numeric invariants	8
1.8 Lattices as orders	9
1.9 Algebras	11
1.10 Lattices as algebras	12
Exercises	15
2 How to Describe Lattices	21
2.1 Lattice diagrams	21
2.2 Join- and meet-tables	21
2.3 Combinations	22
Exercises	24
3 Some Basic Concepts	28
3.1 The concept of isomorphism	28
3.2 Homomorphisms	30
3.3 sublattices and extensions	31
3.4 Ideals	31
3.5 Intervals	35

3.6	Congruences	36
3.7	Congruences and homomorphisms	40
3.8	Congruences and extensions	41
3.9	Congruences and quotients	42
3.10	◇ Tolerances	43
3.11	Direct products	45
3.12	Closure systems	47
3.13	Galois connections	49
3.14	Complete lattices	50
3.15	Algebraic lattices	52
3.16	◇ Continuous lattices by <i>Jimmie D. Lawson</i>	54
3.17	◇ Algebraic lattices in universal algebra	57
	Exercises	59
4	Terms, Identities, and Inequalities	66
4.1	Terms and polynomials	66
4.2	Identities and inequalities	68
4.3	Distributivity and modularity	71
	Exercises	73
5	Free Lattices	75
5.1	The formal definition	75
5.2	Existence	77
5.3	Examples	82
5.4	Partial lattices	83
5.5	Free lattices over partial lattices	89
5.6	◇ Finitely presented lattices	91
	Exercises	92
6	Special Elements	97
6.1	Complements	97
6.2	Pseudocomplements	99
6.3	Other types of special elements	101
6.4	◇ Axiomatic games	102
	Exercises	104
II	Distributive Lattices	109
1	Characterization and Representation Theorems	109
1.1	Characterization theorems	109
1.2	Structure theorems, finite case	112
1.3	◇ Structure theorems, finite case, categorical variant	115
1.4	Structure theorems, infinite case	116
1.5	Some applications	118
1.6	Automorphism groups	120
1.7	◇ Distributive lattices and general algebra	122
	Exercises	123

		126
2	Terms and Freeness	126
	2.1 Terms for distributive lattices	126
	2.2 Boolean terms	128
	2.3 Free constructs	130
	2.4 Boolean homomorphisms	131
	2.5 \diamond Polynomial completeness of lattices by <i>Kalle Kaarti</i>	133
	Exercises	136
3	Congruence Relations	138
	3.1 Principal congruences	138
	3.2 Prime ideals	141
	3.3 Boolean lattices	142
	3.4 Congruence lattices	145
	Exercises	146
4	Boolean Algebras R-generated by Distributive Lattices	149
	4.1 Embedding results	149
	4.2 The complete case	154
	4.3 Boolean lattices generated by chains	156
	Exercises	164
5	Topological Representation	166
	5.1 Distributive join-semilattices	167
	5.2 Stone spaces	168
	5.3 The characterization of Stone spaces	170
	5.4 Applications	175
	5.5 Free distributive products	177
	5.6 \diamond Priestley spaces by <i>Hilary A. Priestley</i>	180
	5.7 \diamond Frames by <i>Aleš Pultr</i>	184
	Exercises	185
6	Pseudocomplementation	191
	6.1 Definitions and examples	191
	6.2 Stone algebras	193
	6.3 Triple construction	194
	6.4 A characterization theorem for Stone algebras	196
	6.5 Two representation theorems for Stone algebras	197
	6.6 \diamond Generalizing Stone algebras	202
	6.7 \diamond Background	202
	Exercises	202
III	Congruences	207
1	Congruence Spreading	207
	1.1 Congruence-perspectivity	207
	1.2 Principal congruences	209
	1.3 The join formula	212
	1.4 Finite lattices	213

1.5	Congruences and extensions	214
1.6	Congruence-preserving extensions	217
1.7	Weakly modular lattices	218
1.8	Representable congruences	219
	Exercises	220
2	Distributive, Standard, and Neutral Elements	223
2.1	The three element types	223
2.2	Distributive elements	223
2.3	Standard elements	224
2.4	Neutral elements	226
2.5	Connections	228
2.6	The set of distributive, standard, and neutral elements	230
	Exercises	232
3	Distributive, Standard, and Neutral Ideals	234
3.1	Defining the three ideal types	234
3.2	Characterization theorems	235
3.3	The associated congruences	238
	Exercises	241
4	Structure Theorems	244
4.1	Direct decompositions	244
4.2	Indecomposable and simple factors	246
4.3	Boolean congruence lattices	248
4.4	◇ Infinite direct decompositions of complete lattices by <i>Friedrich Wehrung</i>	251
	Exercises	252
IV	Lattice Constructions	255
1	Adding an Element	255
1.1	One-Point Extension	255
1.2	Doubling elements and intervals	259
	Exercises	260
2	Gluing	262
2.1	Definition	263
2.2	Congruences	265
2.3	◇ Generalizations	266
	Exercises	267
3	Chopped Lattices	269
3.1	Basic definitions	269
3.2	Compatible vectors of elements	270
3.3	Compatible congruence vectors	272
3.4	From the chopped lattice to the ideal lattice	273
	Exercises	275
4	Constructing Lattices with Given Congruence Lattices	276
4.1	The finite case	276

4.2	Construction and proof	279
4.3	Sectional complementation	280
4.4	◇ Finite lattices by <i>J. B. Nation</i>	282
4.5	◇ Finite lattices in special classes	285
4.6	◇ Two finite lattices	286
4.7	◇ More than two finite lattices	287
4.8	◇ Independence theorem for finite lattices	288
4.9	◇ General lattices	289
4.10	◇ Complete lattices	291
	Exercises	292
5	Boolean Triples	294
5.1	The general construction	295
5.2	Congruence-preserving extension	297
5.3	The distributive case	299
5.4	◇ Tensor products	300
5.5	◇ Congruence-permutable, congruence-preserving extensions by <i>Friedrich Wehrung</i>	301
	Exercises	303
V	Modular and Semimodular Lattices	307
1	Modular Lattices	307
1.1	Equivalent forms	307
1.2	The Isomorphism Theorem for Modular Lattices	308
1.3	Two applications	309
1.4	Congruence spreading	311
1.5	Congruences in the finite case	316
1.6	Von Neumann independence	316
1.7	Sublattice theorems	319
1.8	◇ Pseudocomplemented modular lattices by <i>Tibor Katriňák</i>	321
1.9	◇ Identities and quasi-identities in submodule lattices by <i>Gábor Czédli</i>	323
	Exercises	325
2	Semimodular Lattices	329
2.1	The basic definition	329
2.2	Equivalent formulations	331
2.3	The Jordan-Hölder Theorem	333
2.4	Independence of atoms	334
2.5	M-symmetry	335
2.6	◇ Consistency by <i>Manfred Stern</i>	338
	Exercises	340
3	Geometric Lattices	342
3.1	Definition and basic properties	342
3.2	Structure theorems	344

3.3	Geometries	349
3.4	Graphs	352
3.5	Whitney numbers	353
	Exercises	355
4	Partition Lattices	359
4.1	Basic properties	359
4.2	Type 3 representations	362
4.3	Type 2 representations	365
4.4	Type 1 representations	367
4.5	◇ Type 2 and 3 congruence lattices in algebras	369
4.6	◇ Sublattices of finite partition lattices	370
4.7	◇ Generating partition lattices	371
	Exercises	371
5	Complemented Modular Lattices	373
5.1	Congruences	373
5.2	Modular geometric lattices	373
5.3	Projective spaces	375
5.4	The lattice $PG(D, m)$	378
5.5	Desargues' Theorem	379
5.6	Arguesian lattices	383
5.7	The Coordinatization Theorem	384
5.8	Frink's Embedding Theorem	387
5.9	A weaker version of the arguesian identity	390
5.10	Projective planes	392
5.11	◇ Coordinatizing sectionally complemented modular lattices by <i>Friedrich Wehrung</i>	394
5.12	◇ The dimension monoid of a lattice by <i>Friedrich Wehrung</i>	397
5.13	◇ Dilworth's Covering Theorem by <i>Joseph P. S. Kung</i>	401
	Exercises	403
VI	Varieties of Lattices	409
1	Characterizations of Varieties	409
1.1	Basic definitions and results	409
1.2	Fully invariant congruences	411
1.3	Formulas for $\text{Var}(K)$	412
1.4	Jónsson's Lemma	415
	Exercises	419
2	The Lattice of Varieties of Lattices	423
2.1	Basic properties	423
2.2	◇ Varieties of finite height	425
2.3	Join-irreducible varieties	426
2.4	2^{No} lattice varieties	428
2.5	◇ The covers of the pentagon	429

2.6	◇ Products of varieties	430
2.7	◇ Lattices of equational theories and quasi-equational theories by <i>Kira Adaricheva</i>	431
2.8	◇ Modified Priestley dualities as natural dualities by <i>Brian A. Davey and Miroslav Haviar</i>	434
	Exercises	437
3	Finding Equational Bases	438
3.1	UDE-s and identities	438
3.2	Bounded sequences of intervals	443
3.3	The modular varieties covering M_3	445
	Exercises	450
4	The Amalgamation Property	454
4.1	Basic definitions and elementary results	454
4.2	Lattice varieties with the Amalgamation Property	458
4.3	The class $\text{Amal}(K)$	461
	Exercises	464
VII Free Products		467
1	Free Products of Lattices	467
1.1	Introduction	467
1.2	The basic definitions	470
1.3	Covers	471
1.4	The algorithm	472
1.5	Computing the algorithm	472
1.6	Representing the free product	474
1.7	The Structure Theorem for Free Products	476
1.8	Sublattices of a free product satisfying (W)	480
1.9	Minimal representations	481
1.10	Sublattices of a free product satisfying (SD_\vee)	484
1.11	The Common Refinement Property	485
1.12	◇ Bounded and amalgamated free products	487
1.13	◇ Distributive free products	488
	Exercises	488
2	The Structure of Free Lattices	493
2.1	The structure theorem	493
2.2	◇ The word problem for modular lattices	494
2.3	Applications	494
2.4	Sublattices	498
2.5	◇ More covers	501
2.6	◇ Finite sublattices and transferable lattices	502
2.7	◇ Semidistributive lattices by <i>Kira Adaricheva</i>	503
	Exercises	506
3	Reduced Free Products	508
3.1	Basic definitions	508

3.2	The structure theorem	508
3.3	Getting ready for applications	511
3.4	Embedding into uniquely complemented lattices	514
3.5	◊ Dean's Lemma	517
3.6	Some applications of Dean's Lemma	518
	Exercises	521
4	Hopfian Lattices	526
4.1	Basic definitions	526
4.2	Free product of hopfian lattices	528
	Exercises	531
	Afterword	533
	Bibliography	539
	Index	589