

# Contents

---

## Part I. Introduction

---

<b>Prologue</b> .....	1
<b>1. Mathematical Principles of Modern Natural Philosophy</b> .	11
1.1 Basic Principles .....	12
1.2 The Infinitesimal Strategy and Differential Equations.....	14
1.3 The Optimality Principle .....	14
1.4 The Basic Notion of Action in Physics and the Idea of Quantization .....	15
1.5 The Method of the Green's Function .....	17
1.6 Harmonic Analysis and the Fourier Method .....	21
1.7 The Method of Averaging and the Theory of Distributions .	26
1.8 The Symbolic Method .....	28
1.9 Gauge Theory – Local Symmetry and the Description of Interactions by Gauge Fields .....	34
1.10 The Challenge of Dark Matter.....	46
<b>2. The Basic Strategy of Extracting Finite Information from   Infinites – Ariadne's Thread in Renormalization Theory</b>	47
2.1 Renormalization Theory in a Nutshell .....	48
2.1.1 Effective Frequency and Running Coupling Constant of an Anharmonic Oscillator .....	48
2.1.2 The Zeta Function and Riemann's Idea of Analytic Continuation .....	54
2.1.3 Meromorphic Functions and Mittag-Leffler's Idea of Subtractions .....	56
2.1.4 The Square of the Dirac Delta Function.....	58
2.2 Regularization of Divergent Integrals in Baby Renormalization Theory .....	60
2.2.1 Momentum Cut-off and the Method of Power-Counting.....	60
2.2.2 The Choice of the Normalization Momentum .....	63
2.2.3 The Method of Differentiating Parameter Integrals .	63
2.2.4 The Method of Taylor Subtraction .....	64

2.2.5	Overlapping Divergences . . . . .	65
2.2.6	The Role of Counterterms . . . . .	67
2.2.7	Euler's Gamma Function . . . . .	67
2.2.8	Integration Tricks . . . . .	69
2.2.9	Dimensional Regularization via Analytic Continuation . . . . .	73
2.2.10	Pauli–Villars Regularization . . . . .	76
2.2.11	Analytic Regularization . . . . .	77
2.2.12	Application to Algebraic Feynman Integrals in Minkowski Space . . . . .	80
2.2.13	Distribution-Valued Meromorphic Functions . . . . .	81
2.2.14	Application to Newton's Equation of Motion . . . . .	87
2.2.15	Hints for Further Reading. . . . .	92
2.3	Further Regularization Methods in Mathematics . . . . .	93
2.3.1	Euler's Philosophy . . . . .	93
2.3.2	Adiabatic Regularization of Divergent Series . . . . .	94
2.3.3	Adiabatic Regularization of Oscillating Integrals . . . . .	95
2.3.4	Regularization by Averaging . . . . .	96
2.3.5	Borel Regularization . . . . .	98
2.3.6	Hadamard's Finite Part of Divergent Integrals . . . . .	100
2.3.7	Infinite-Dimensional Gaussian Integrals and the Zeta Function Regularization . . . . .	101
2.4	Trouble in Mathematics . . . . .	102
2.4.1	Interchanging Limits . . . . .	102
2.4.2	The Ambiguity of Regularization Methods . . . . .	104
2.4.3	Pseudo-Convergence . . . . .	104
2.4.4	Ill-Posed Problems . . . . .	105
2.5	Mathemagics . . . . .	109
<b>3.</b>	<b>The Power of Combinatorics . . . . .</b>	<b>115</b>
3.1	Algebras . . . . .	115
3.2	The Algebra of Multilinear Functionals . . . . .	117
3.3	Fusion, Splitting, and Hopf Algebras . . . . .	122
3.3.1	The Bialgebra of Linear Differential Operators . . . . .	123
3.3.2	The Definition of Hopf Algebras . . . . .	128
3.4	Power Series Expansion and Hopf Algebras . . . . .	131
3.4.1	The Importance of Cancellations . . . . .	131
3.4.2	The Kepler Equation and the Lagrange Inversion Formula . . . . .	132
3.4.3	The Composition Formula for Power Series . . . . .	134
3.4.4	The Faà di Bruno Hopf Algebra for the Formal Diffeomorphism Group of the Complex Plane . . . . .	136
3.4.5	The Generalized Zimmermann Forest Formula . . . . .	138
3.4.6	The Logarithmic Function and Schur Polynomials . . . . .	140
3.4.7	Correlation Functions in Quantum Field Theory . . . . .	141

3.4.8	Random Variables, Moments, and Cumulants . . . . .	143
3.5	Symmetry and Hopf Algebras . . . . .	146
3.5.1	The Strategy of Coordinatization in Mathematics and Physics . . . . .	146
3.5.2	The Coordinate Hopf Algebra of a Finite Group . . .	148
3.5.3	The Coordinate Hopf Algebra of an Operator Group	150
3.5.4	The Tannaka–Krein Duality for Compact Lie Groups	152
3.6	Regularization and Rota–Baxter Algebras . . . . .	154
3.6.1	Regularization of the Laurent Series . . . . .	157
3.6.2	Projection Operators . . . . .	158
3.6.3	The $q$ -Integral . . . . .	158
3.6.4	The Volterra–Spitzer Exponential Formula . . . . .	160
3.6.5	The Importance of the Exponential Function in Mathematics and Physics . . . . .	161
3.7	Partially Ordered Sets and Combinatorics . . . . .	162
3.7.1	Incidence Algebras and the Zeta Function . . . . .	162
3.7.2	The Möbius Function as an Inverse Function . . . . .	163
3.7.3	The Inclusion–Exclusion Principle in Combinatorics	164
3.7.4	Applications to Number Theory . . . . .	166
3.8	Hints for Further Reading . . . . .	167
<b>4.</b>	<b>The Strategy of Equivalence Classes in Mathematics . . . . .</b>	<b>175</b>
4.1	Equivalence Classes in Algebra . . . . .	178
4.1.1	The Gaussian Quotient Ring and the Quadratic Reciprocity Law in Number Theory . . . . .	178
4.1.2	Application of the Fermat–Euler Theorem in Coding Theory . . . . .	182
4.1.3	Quotient Rings, Quotient Groups, and Quotient Fields	184
4.1.4	Linear Quotient Spaces . . . . .	188
4.1.5	Ideals and Quotient Algebras . . . . .	190
4.2	Superfunctions and the Heaviside Calculus in Electrical Engineering . . . . .	191
4.3	Equivalence Classes in Geometry . . . . .	194
4.3.1	The Basic Idea of Geometry Epitomized by Klein’s Erlangen Program . . . . .	194
4.3.2	Symmetry Spaces, Orbit Spaces, and Homogeneous Spaces . . . . .	194
4.3.3	The Space of Quantum States . . . . .	199
4.3.4	Real Projective Spaces . . . . .	200
4.3.5	Complex Projective Spaces . . . . .	203
4.3.6	The Shape of the Universe . . . . .	204
4.4	Equivalence Classes in Topology . . . . .	205
4.4.1	Topological Quotient Spaces . . . . .	205
4.4.2	Physical Fields, Observers, Bundles, and Cocycles . .	208
4.4.3	Generalized Physical Fields and Sheaves . . . . .	216

4.4.4	Deformations, Mapping Classes, and Topological Charges . . . . .	219
4.4.5	Poincaré's Fundamental Group . . . . .	223
4.4.6	Loop Spaces and Higher Homotopy Groups . . . . .	225
4.4.7	Homology, Cohomology, and Electrodynamics . . . . .	227
4.4.8	Bott's Periodicity Theorem . . . . .	227
4.4.9	$K$ -Theory . . . . .	228
4.4.10	Application to Fredholm Operators . . . . .	233
4.4.11	Hints for Further Reading . . . . .	235
4.5	The Strategy of Partial Ordering . . . . .	237
4.5.1	Feynman Diagrams . . . . .	238
4.5.2	The Abstract Entropy Principle in Thermodynamics . . . . .	239
4.5.3	Convergence of Generalized Sequences . . . . .	240
4.5.4	Inductive and Projective Topologies . . . . .	241
4.5.5	Inductive and Projective Limits . . . . .	243
4.5.6	Classes, Sets, and Non-Sets . . . . .	245
4.5.7	The Fixed-Point Theorem of Bourbaki-Kneser . . . . .	247
4.5.8	Zorn's Lemma . . . . .	248
4.6	Leibniz's Infinitesimals and Non-Standard Analysis . . . . .	248
4.6.1	Filters and Ultrafilters . . . . .	250
4.6.2	The Full-Rigged Real Line . . . . .	251

---

## Part II. Basic Ideas in Classical Mechanics

---

5.	Geometrical Optics . . . . .	263
5.1	Ariadne's Thread in Geometrical Optics . . . . .	264
5.2	Fermat's Principle of Least Time . . . . .	268
5.3	Huygens' Principle on Wave Fronts . . . . .	270
5.4	Carathéodory's Royal Road to Geometrical Optics . . . . .	271
5.5	The Duality between Light Rays and Wave Fronts . . . . .	274
5.5.1	From Wave Fronts to Light Rays . . . . .	275
5.5.2	From Light Rays to Wave Fronts . . . . .	276
5.6	The Jacobi Approach to Focal Points . . . . .	276
5.7	Lie's Contact Geometry . . . . .	279
5.7.1	Basic Ideas . . . . .	279
5.7.2	Contact Manifolds and Contact Transformations . . . . .	283
5.7.3	Applications to Geometrical Optics . . . . .	284
5.7.4	Equilibrium Thermodynamics and Legendre Submanifolds . . . . .	285
5.8	Light Rays and Non-Euclidean Geometry . . . . .	289
5.8.1	Linear Symplectic Spaces . . . . .	291
5.8.2	The Kähler Form of a Complex Hilbert Space . . . . .	295
5.8.3	The Refraction Index and Geodesics . . . . .	297
5.8.4	The Trick of Gauge Fixing . . . . .	299

5.8.5	Geodesic Flow . . . . .	299
5.8.6	Hamilton's Duality Trick and Cogeodesic Flow . . . . .	300
5.8.7	The Principle of Minimal Geodesic Energy . . . . .	301
5.9	Spherical Geometry . . . . .	302
5.9.1	The Global Gauss–Bonnet Theorem . . . . .	303
5.9.2	De Rham Cohomology and the Chern Class of the Sphere . . . . .	305
5.9.3	The Beltrami Model . . . . .	308
5.10	The Poincaré Model of Hyperbolic Geometry . . . . .	314
5.10.1	Kähler Geometry and the Gaussian Curvature . . . . .	318
5.10.2	Kähler–Einstein Geometry . . . . .	323
5.10.3	Symplectic Geometry . . . . .	323
5.10.4	Riemannian Geometry . . . . .	324
5.11	Ariadne's Thread in Gauge Theory . . . . .	333
5.11.1	Parallel Transport of Physical Information – the Key to Modern Physics . . . . .	334
5.11.2	The Phase Equation and Fiber Bundles . . . . .	337
5.11.3	Gauge Transformations and Gauge-Invariant Differential Forms . . . . .	338
5.11.4	Perspectives . . . . .	341
5.12	Classification of Two-Dimensional Compact Manifolds . . . . .	343
5.13	The Poincaré Conjecture and the Ricci Flow . . . . .	346
5.14	A Glance at Modern Optimization Theory . . . . .	348
5.15	Hints for Further Reading . . . . .	348

**6. The Principle of Critical Action and the Harmonic**

	<b>Oscillator – Ariadne's Thread in Classical Mechanics . . . . .</b>	<b>359</b>
6.1	Prototypes of Extremal Problems . . . . .	360
6.2	The Motion of a Particle . . . . .	364
6.3	Newtonian Mechanics . . . . .	366
6.4	A Glance at the History of the Calculus of Variations . . . . .	370
6.5	Lagrangian Mechanics . . . . .	372
6.5.1	The Harmonic Oscillator . . . . .	373
6.5.2	The Euler–Lagrange Equation . . . . .	375
6.5.3	Jacobi's Accessory Eigenvalue Problem . . . . .	376
6.5.4	The Morse Index . . . . .	377
6.5.5	The Anharmonic Oscillator . . . . .	378
6.5.6	The Ginzburg–Landau Potential and the Higgs Potential . . . . .	380
6.5.7	Damped Oscillations, Stability, and Energy Dissipation . . . . .	382
6.5.8	Resonance and Small Divisors . . . . .	382
6.6	Symmetry and Conservation Laws . . . . .	383
6.6.1	The Symmetries of the Harmonic Oscillator . . . . .	384
6.6.2	The Noether Theorem . . . . .	384

6.7	The Pendulum and Dynamical Systems . . . . .	390
6.7.1	The Equation of Motion . . . . .	390
6.7.2	Elliptic Integrals and Elliptic Functions . . . . .	391
6.7.3	The Phase Space of the Pendulum and Bundles . . . .	396
6.8	Hamiltonian Mechanics . . . . .	402
6.8.1	The Canonical Equation . . . . .	404
6.8.2	The Hamiltonian Flow . . . . .	404
6.8.3	The Hamilton–Jacobi Partial Differential Equation .	405
6.9	Poissonian Mechanics . . . . .	406
6.9.1	Poisson Brackets and the Equation of Motion . . . . .	407
6.9.2	Conservation Laws . . . . .	407
6.10	Symplectic Geometry . . . . .	407
6.10.1	The Canonical Equations . . . . .	408
6.10.2	Symplectic Transformations . . . . .	409
6.11	The Spherical Pendulum . . . . .	411
6.11.1	The Gaussian Principle of Critical Constraint . . . . .	411
6.11.2	The Lagrangian Approach . . . . .	412
6.11.3	The Hamiltonian Approach . . . . .	414
6.11.4	Geodesics of Shortest Length . . . . .	415
6.12	The Lie Group $SU(E^3)$ of Rotations . . . . .	416
6.12.1	Conservation of Angular Momentum . . . . .	416
6.12.2	Lie’s Momentum Map . . . . .	419
6.13	Carathéodory’s Royal Road to the Calculus of Variations . .	419
6.13.1	The Fundamental Equation . . . . .	419
6.13.2	Lagrangian Submanifolds in Symplectic Geometry . .	421
6.13.3	The Initial-Value Problem for the Hamilton–Jacobi Equation . . . . .	423
6.13.4	Solution of Carathéodory’s Fundamental Equation .	423
6.14	Hints for Further Reading . . . . .	424

---

## Part III. Basic Ideas in Quantum Mechanics

---

7.	<b>Quantization of the Harmonic Oscillator – Ariadne’s Thread in Quantization</b> . . . . .	427
7.1	Complete Orthonormal Systems . . . . .	430
7.2	Bosonic Creation and Annihilation Operators . . . . .	432
7.3	Heisenberg’s Quantum Mechanics . . . . .	440
7.3.1	Heisenberg’s Equation of Motion . . . . .	443
7.3.2	Heisenberg’s Uncertainty Inequality for the Harmonic Oscillator . . . . .	446
7.3.3	Quantization of Energy . . . . .	447
7.3.4	The Transition Probabilities . . . . .	449
7.3.5	The Wightman Functions . . . . .	451
7.3.6	The Correlation Functions . . . . .	456

7.4	Schrödinger's Quantum Mechanics . . . . .	459
7.4.1	The Schrödinger Equation . . . . .	459
7.4.2	States, Observables, and Measurements . . . . .	462
7.4.3	The Free Motion of a Quantum Particle . . . . .	464
7.4.4	The Harmonic Oscillator . . . . .	467
7.4.5	The Passage to the Heisenberg Picture . . . . .	473
7.4.6	Heisenberg's Uncertainty Principle . . . . .	475
7.4.7	Unstable Quantum States and the Energy-Time Uncertainty Relation . . . . .	476
7.4.8	Schrödinger's Coherent States . . . . .	478
7.5	Feynman's Quantum Mechanics . . . . .	479
7.5.1	Main Ideas . . . . .	480
7.5.2	The Diffusion Kernel and the Euclidean Strategy in Quantum Physics . . . . .	487
7.5.3	Probability Amplitudes and the Formal Propagator Theory . . . . .	488
7.6	Von Neumann's Rigorous Approach . . . . .	495
7.6.1	The Prototype of the Operator Calculus . . . . .	496
7.6.2	The General Operator Calculus . . . . .	499
7.6.3	Rigorous Propagator Theory . . . . .	505
7.6.4	The Free Quantum Particle as a Paradigm of Functional Analysis . . . . .	509
7.6.5	The Free Hamiltonian . . . . .	524
7.6.6	The Rescaled Fourier Transform . . . . .	532
7.6.7	The Quantized Harmonic Oscillator and the Maslov Index . . . . .	534
7.6.8	Ideal Gases and von Neumann's Density Operator . .	540
7.7	The Feynman Path Integral . . . . .	547
7.7.1	The Basic Strategy . . . . .	547
7.7.2	The Basic Definition . . . . .	549
7.7.3	Application to the Free Quantum Particle . . . . .	550
7.7.4	Application to the Harmonic Oscillator . . . . .	552
7.7.5	The Propagator Hypothesis . . . . .	555
7.7.6	Motivation of Feynman's Path Integral . . . . .	555
7.8	Finite-Dimensional Gaussian Integrals . . . . .	559
7.8.1	Basic Formulas . . . . .	560
7.8.2	Free Moments, the Wick Theorem, and Feynman Diagrams . . . . .	564
7.8.3	Full Moments and Perturbation Theory . . . . .	567
7.9	Rigorous Infinite-Dimensional Gaussian Integrals . . . . .	570
7.9.1	The Infinite-Dimensional Dispersion Operator . . . .	571
7.9.2	Zeta Function Regularization and Infinite-Dimensional Determinants . . . . .	572
7.9.3	Application to the Free Quantum Particle . . . . .	574
7.9.4	Application to the Quantized Harmonic Oscillator . .	576

7.9.5	The Spectral Hypothesis .....	579
7.10	The Semi-Classical WKB Method .....	580
7.11	Brownian Motion .....	584
7.11.1	The Macroscopic Diffusion Law .....	584
7.11.2	Einstein's Key Formulas for the Brownian Motion ..	585
7.11.3	The Random Walk of Particles .....	585
7.11.4	The Rigorous Wiener Path Integral .....	586
7.11.5	The Feynman–Kac Formula .....	588
7.12	Weyl Quantization .....	590
7.12.1	The Formal Moyal Star Product .....	591
7.12.2	Deformation Quantization of the Harmonic Oscillator	592
7.12.3	Weyl Ordering .....	596
7.12.4	Operator Kernels .....	599
7.12.5	The Formal Weyl Calculus .....	602
7.12.6	The Rigorous Weyl Calculus .....	606
7.13	Two Magic Formulas .....	608
7.13.1	The Formal Feynman Path Integral for the Propaga- tor Kernel .....	611
7.13.2	The Relation between the Scattering Kernel and the Propagator Kernel .....	614
7.14	The Poincaré–Wirtinger Calculus .....	616
7.15	Bargmann's Holomorphic Quantization .....	617
7.16	The Stone–Von Neumann Uniqueness Theorem .....	621
7.16.1	The Prototype of the Weyl Relation .....	621
7.16.2	The Main Theorem .....	626
7.16.3	$C^*$ -Algebras .....	627
7.16.4	Operator Ideals .....	629
7.16.5	Symplectic Geometry and the Weyl Quantization Functor .....	630
7.17	A Glance at the Algebraic Approach to Quantum Physics ..	633
7.17.1	States and Observables .....	633
7.17.2	Gleason's Extension Theorem – the Main Theorem of Quantum Logic .....	637
7.17.3	The Finite Standard Model in Statistical Physics as a Paradigm .....	638
7.17.4	Information, Entropy, and the Measure of Disorder .	640
7.17.5	Semiclassical Statistical Physics .....	645
7.17.6	The Classical Ideal Gas .....	648
7.17.7	Bose–Einstein Statistics .....	649
7.17.8	Fermi–Dirac Statistics .....	650
7.17.9	Thermodynamic Equilibrium and KMS-States .....	651
7.17.10	Quasi-Stationary Thermodynamic Processes and Irreversibility .....	652
7.17.11	The Photon Mill on Earth .....	654
7.18	Von Neumann Algebras .....	654



7.18.1	Von Neumann's Bicommutant Theorem . . . . .	655
7.18.2	The Murray–von Neumann Classification of Factors . . . . .	658
7.18.3	The Tomita–Takesaki Theory and KMS-States . . . . .	659
7.19	Connes' Noncommutative Geometry . . . . .	660
7.20	Jordan Algebras . . . . .	662
7.21	The Supersymmetric Harmonic Oscillator . . . . .	663
7.22	Hints for Further Reading . . . . .	667
<b>8.</b>	<b>Quantum Particles on the Real Line – Ariadne's Thread in Scattering Theory . . . . .</b>	<b>699</b>
8.1	Classical Dynamics Versus Quantum Dynamics . . . . .	699
8.2	The Stationary Schrödinger Equation . . . . .	703
8.3	One-Dimensional Quantum Motion in a Square-Well Potential . . . . .	704
8.3.1	Free Motion . . . . .	704
8.3.2	Scattering States and the $S$ -Matrix . . . . .	705
8.3.3	Bound States . . . . .	710
8.3.4	Bound-State Energies and the Singularities of the $S$ -Matrix . . . . .	712
8.3.5	The Energetic Riemann Surface, Resonances, and the Breit–Wigner Formula . . . . .	713
8.3.6	The Jost Functions . . . . .	718
8.3.7	The Fourier–Stieltjes Transformation . . . . .	719
8.3.8	Generalized Eigenfunctions of the Hamiltonian . . . . .	720
8.3.9	Quantum Dynamics and the Scattering Operator . . . . .	722
8.3.10	The Feynman Propagator . . . . .	726
8.4	Tunnelling of Quantum Particles and Radioactive Decay . . . . .	727
8.5	The Method of the Green's Function in a Nutshell . . . . .	729
8.5.1	The Inhomogeneous Helmholtz Equation . . . . .	730
8.5.2	The Retarded Green's Function, and the Existence and Uniqueness Theorem . . . . .	731
8.5.3	The Advanced Green's Function . . . . .	736
8.5.4	Perturbation of the Retarded and Advanced Green's Function . . . . .	737
8.5.5	Feynman's Regularized Fourier Method . . . . .	739
8.6	The Lippmann–Schwinger Integral Equation . . . . .	743
8.6.1	The Born Approximation . . . . .	743
8.6.2	The Existence and Uniqueness Theorem via Banach's Fixed Point Theorem . . . . .	744
8.6.3	Hypoellipticity . . . . .	745
<b>9.</b>	<b>A Glance at General Scattering Theory . . . . .</b>	<b>747</b>
9.1	The Formal Basic Idea . . . . .	749
9.2	The Rigorous Time-Dependent Approach . . . . .	751
9.3	The Rigorous Time-Independent Approach . . . . .	753

9.4	Applications to Quantum Mechanics .....	754
9.5	A Glance at Quantum Field Theory .....	757
9.6	Hints for Further Reading .....	758

---

## Part IV. Quantum Electrodynamics (QED)

---

<b>10.</b>	<b>Creation and Annihilation Operators .....</b>	<b>771</b>
10.1	The Bosonic Fock Space .....	771
10.1.1	The Particle Number Operator .....	774
10.1.2	The Ground State .....	774
10.2	The Fermionic Fock Space and the Pauli Principle .....	779
10.3	General Construction .....	784
10.4	The Main Strategy of Quantum Electrodynamics .....	788
<b>11.</b>	<b>The Basic Equations in Quantum Electrodynamics .....</b>	<b>793</b>
11.1	The Classical Lagrangian .....	793
11.2	The Gauge Condition .....	796
<b>12.</b>	<b>The Free Quantum Fields of Electrons, Positrons, and Photons .....</b>	<b>799</b>
12.1	Classical Free Fields .....	799
12.1.1	The Lattice Strategy in Quantum Electrodynamics .	799
12.1.2	The High-Energy Limit and the Low-Energy Limit .	802
12.1.3	The Free Electromagnetic Field .....	803
12.1.4	The Free Electron Field .....	806
12.2	Quantization .....	811
12.2.1	The Free Photon Quantum Field .....	812
12.2.2	The Free Electron Quantum Field and Antiparticles	814
12.2.3	The Spin of Photons .....	819
12.3	The Ground State Energy and the Normal Product .....	822
12.4	The Importance of Mathematical Models .....	824
12.4.1	The Trouble with Virtual Photons .....	825
12.4.2	Indefinite Inner Product Spaces .....	826
12.4.3	Representation of the Creation and Annihilation Operators in QED .....	826
12.4.4	Gupta-Bleuler Quantization .....	831
<b>13.</b>	<b>The Interacting Quantum Field, and the Magic Dyson Series for the <math>S</math>-Matrix .....</b>	<b>835</b>
13.1	Dyson's Key Formula .....	835
13.2	The Basic Strategy of Reduction Formulas .....	841
13.3	The Wick Theorem .....	846
13.4	Feynman Propagators .....	856

13.4.1	Discrete Feynman Propagators for Photons and Electrons . . . . .	856
13.4.2	Regularized Discrete Propagators . . . . .	862
13.4.3	The Continuum Limit of Feynman Propagators . . . . .	864
13.4.4	Classical Wave Propagation versus Feynman Propagator . . . . .	870
<b>14.</b>	<b>The Beauty of Feynman Diagrams in QED . . . . .</b>	<b>875</b>
14.1	Compton Effect and Feynman Rules in Position Space . . . . .	876
14.2	Symmetry Properties . . . . .	881
14.3	Summary of the Feynman Rules in Momentum Space . . . . .	882
14.4	Typical Examples . . . . .	885
14.5	The Formal Language of Physicists . . . . .	890
14.6	Transition Probabilities and Cross Sections of Scattering Processes . . . . .	891
14.7	The Crucial Limits . . . . .	894
14.8	Appendix: Table of Feynman Rules . . . . .	896
<b>15.</b>	<b>Applications to Physical Effects . . . . .</b>	<b>899</b>
15.1	Compton Effect . . . . .	899
15.1.1	Duality between Light Waves and Light Particles in the History of Physics . . . . .	902
15.1.2	The Trace Method for Computing Cross Sections . . . . .	903
15.1.3	Relativistic Invariance . . . . .	912
15.2	Asymptotically Free Electrons in an External Electromagnetic Field . . . . .	914
15.2.1	The Key Formula for the Cross Section . . . . .	914
15.2.2	Application to Yukawa Scattering . . . . .	915
15.2.3	Application to Coulomb Scattering . . . . .	915
15.2.4	Motivation of the Key Formula via $S$ -Matrix . . . . .	916
15.2.5	Perspectives . . . . .	921
15.3	Bound Electrons in an External Electromagnetic Field . . . . .	922
15.3.1	The Spontaneous Emission of Photons by the Atom . . . . .	922
15.3.2	Motivation of the Key Formula . . . . .	923
15.3.3	Intensity of Spectral Lines . . . . .	925
15.4	Cherenkov Radiation . . . . .	926

---

## Part V. Renormalization

---

<b>16.</b>	<b>The Continuum Limit . . . . .</b>	<b>945</b>
16.1	The Fundamental Limits . . . . .	945
16.2	The Formal Limits Fail . . . . .	946
16.3	Basic Ideas of Renormalization . . . . .	947

16.3.1	The Effective Mass and the Effective Charge of the Electron . . . . .	947
16.3.2	The Counterterms of the Modified Lagrangian . . . . .	947
16.3.3	The Compensation Principle . . . . .	948
16.3.4	Fundamental Invariance Principles . . . . .	949
16.3.5	Dimensional Regularization of Discrete Algebraic Feynman Integrals . . . . .	949
16.3.6	Multiplicative Renormalization . . . . .	950
16.4	The Theory of Approximation Schemes in Mathematics . . . . .	951
<b>17.</b>	<b>Radiative Corrections of Lowest Order . . . . .</b>	<b>953</b>
17.1	Primitive Divergent Feynman Graphs . . . . .	953
17.2	Vacuum Polarization . . . . .	954
17.3	Radiative Corrections of the Propagators . . . . .	955
17.3.1	The Photon Propagator . . . . .	956
17.3.2	The Electron Propagator . . . . .	956
17.3.3	The Vertex Correction and the Ward Identity . . . . .	957
17.4	The Counterterms of the Lagrangian and the Compensation Principle . . . . .	957
17.5	Application to Physical Problems . . . . .	958
17.5.1	Radiative Correction of the Coulomb Potential . . . . .	958
17.5.2	The Anomalous Magnetic Moment of the Electron . . . . .	959
17.5.3	The Anomalous Magnetic Moment of the Muon . . . . .	961
17.5.4	The Lamb Shift . . . . .	962
17.5.5	Photon-Photon Scattering . . . . .	964
<b>18.</b>	<b>A Glance at Renormalization to all Orders of Perturbation Theory . . . . .</b>	<b>967</b>
18.1	One-Particle Irreducible Feynman Graphs and Divergences . . . . .	970
18.2	Overlapping Divergences and Manoukian's Equivalence Principle . . . . .	972
18.3	The Renormalizability of Quantum Electrodynamics . . . . .	975
18.4	Automated Multi-Loop Computations in Perturbation Theory . . . . .	977
<b>19.</b>	<b>Perspectives . . . . .</b>	<b>979</b>
19.1	BPHZ Renormalization . . . . .	981
19.1.1	Bogoliubov's Iterative <i>R</i> -Method . . . . .	981
19.1.2	Zimmermann's Forest Formula . . . . .	984
19.1.3	The Classical BPHZ Method . . . . .	985
19.2	The Causal Epstein-Glaser <i>S</i> -Matrix Approach . . . . .	987
19.3	Kreimer's Hopf Algebra Revolution . . . . .	990
19.3.1	The History of the Hopf Algebra Approach . . . . .	991

19.3.2	Renormalization and the Iterative Birkhoff Factorization for Complex Lie Groups . . . . .	993
19.3.3	The Renormalization of Quantum Electrodynamics . . . . .	996
19.4	The Scope of the Riemann–Hilbert Problem . . . . .	997
19.4.1	The Gaussian Hypergeometric Differential Equation . . . . .	998
19.4.2	The Confluent Hypergeometric Function and the Spectrum of the Hydrogen Atom . . . . .	1004
19.4.3	Hilbert’s 21th Problem . . . . .	1004
19.4.4	The Transport of Information in Nature . . . . .	1007
19.4.5	Stable Transport of Energy and Solitons . . . . .	1007
19.4.6	Ariadne’s Thread in Soliton Theory . . . . .	1009
19.4.7	Resonances . . . . .	1014
19.4.8	The Role of Integrable Systems in Nature . . . . .	1014
19.5	The BFFO Hopf Superalgebra Approach . . . . .	1016
19.6	The BRST Approach and Algebraic Renormalization . . . . .	1019
19.7	Analytic Renormalization and Distribution-Valued Analytic Functions . . . . .	1022
19.8	Computational Strategies . . . . .	1023
19.8.1	The Renormalization Group . . . . .	1023
19.8.2	Operator Product Expansions . . . . .	1024
19.8.3	Binary Planar Graphs and the Renormalization of Quantum Electrodynamics . . . . .	1026
19.9	The Master Ward Identity . . . . .	1027
19.10	Trouble in Quantum Electrodynamics . . . . .	1027
19.10.1	The Landau Inconsistency Problem in Quantum Electrodynamics . . . . .	1027
19.10.2	The Lack of Asymptotic Freedom in Quantum Electrodynamics . . . . .	1029
19.11	Hints for Further Reading . . . . .	1029
<b>Epilogue</b> . . . . .		1045
<b>References</b> . . . . .		1049
<b>List of Symbols</b> . . . . .		1061
<b>Index</b> . . . . .		1069