

Contents

1	An Application of the Hard Lefschetz Theorem	1
1.1	Review of Eichler Integrals	1
1.2	Automorphic Representations	6
1.3	CAP and Toric Representations	9
1.4	The l -Adic Sheaves	11
1.5	Lefschetz Maps	12
1.6	Weak Ramanujan	14
1.7	Residues of L -Series	15
2	CAP Localization	19
2.1	Standard Parabolic Subgroups	21
2.2	The Adelic Reductive Borel–Serre Compactification	23
2.2.1	Components	24
2.2.2	Fixed Components h	24
2.2.3	Another Formulation	25
2.2.4	Small Groups	26
2.3	Fixed Points	27
2.3.1	Archimedean Place	28
2.3.2	Non-Archimedean Places	29
2.3.3	Globally	29
2.4	Lefschetz Numbers	30
2.4.1	Euler Characteristics	30
2.4.2	Computation of O_γ	32
2.4.3	Conclusion	32
2.5	Computation of an Orbital Integral	34
2.6	Elliptic Traces	36
2.7	The Satake Transform	38
2.7.1	Subdivision of the Weyl Chambers	42
2.7.2	Global Situation	43
2.8	Automorphic Representations	44
2.9	The Discrete Series Case	48

Appendix 1	50
Appendix 2	51
3 The Ramanujan Conjecture for Genus 2 Siegel Modular Forms	53
3.1 Results Obtained by Kottwitz, Milne, Pink, and Shpiz	56
3.1.1 Stabilization	58
3.2 Destabilization	60
3.3 The Topological Trace Formula	62
3.4 (G, M) -Regularity Removed	63
3.5 CAP Localization Revisited	65
3.6 The Fundamental Lemma	69
3.6.1 Odds and Ends	72
4 Character Identities and Galois Representations Related to the Group $GSp(4)$	75
4.1 Technical Preliminaries	76
4.1.1 Representations Attached to σ and S and p	77
4.2 Galois Representations	78
4.2.1 The Classical λ -Adic Representations	79
4.2.2 Weak Reciprocity Law	80
4.2.3 CM Representations	81
4.3 The Trace Formula Applied	82
4.4 Rationality	89
4.4.1 Character Transfer	91
4.4.2 The List of Maximal F -Tori of G	94
4.4.3 Concerning \mathbb{Q}_l -Rationality	97
4.5 Orthogonality Relations	99
4.5.1 Elliptic Scalar Products	102
4.5.2 The Formula for α^G	102
4.5.3 Induced Representations	106
4.5.4 Scalar Product Formula	107
4.5.5 Character Formula	108
4.6 Exponents and Tempered Representations	109
4.6.1 The Asymptotic Behavior	114
4.6.2 The Theorem of Deligne and Casselman	115
4.7 The Classification of Local Representations	118
4.7.1 The Siegel Parabolic P (Results Obtained by Shahidi)	119
4.7.2 The Klingen Parabolic Q (Results Obtained by Waldspurger)	120
4.7.3 The Borel Group B (Results Obtained by Tadic and Rodier)	121
4.7.4 The Borel Group B (The Nonregular Reducible Principal Series)	124
4.7.5 Results Obtained by Kudla and Rallis	130

4.8	Summary	133
4.8.1	List of Irreducible Unitary Discrete Series Representations of G	135
4.9	Asymptotics	136
4.9.1	The Character Lift	137
4.10	Whittaker Models	144
4.10.1	Theta Lifts	145
4.10.2	The Local Theta Lift	146
4.10.3	The Local L -packets	149
4.11	The Endoscopic Character Lift	151
4.12	The Anisotropic Endoscopic Theta Lift	158
4.12.1	The Global Situation	161
4.12.2	The Anisotropic Theta Lift (Local Theory)	162
4.12.3	The Jacquet–Langlands Lift	163
4.12.4	The Intertwining Map b	164
4.12.5	The Q -Jacquet Module of the Anisotropic Weil Representation $\Theta_-(\sigma)$	166
4.12.6	Whittaker Models	168
4.12.7	The Siegel Parabolic	169
4.12.8	The Anisotropic Theta Lift $\theta_-(\sigma)$ (Global Theory)	171
5	Local and Global Endoscopy for $GSp(4)$	175
5.1	The Local Endoscopic Lift	175
5.1.1	The Integral Character Lift	177
5.1.2	Steps (0)–(xi) of the Proof (in the p -Adic Case)	178
5.1.3	Complements on the Local Theta Lift	181
5.2	The Global Situation	184
5.3	The Multiplicity Formula	191
5.4	Local and Global Trace Identities	197
5.5	Appendix on Arthur’s Trace Formula	203
6	A Special Case of the Fundamental Lemma I	211
6.1	Introduction	211
6.2	The Torus T	213
6.2.1	Orbital Integrals	215
6.3	The Endoscopic Group M	215
6.4	Orbital Integrals for M	219
6.5	Reduction to H	221
6.6	Preliminary Considerations for (x, x')	222
6.7	The Residue Rings $R = o_F/\pi_F^i o_F$	224
6.8	On $T \setminus H$ -Integration	227
6.9	Double Cosets in H	229
6.10	Summation Conditions	230
6.11	Résumé	232
6.12	The Summation (L/F -Ramified)	233

6.13	The Summation (L/F -Unramified)	234
6.14	Appendix on Orders (L/F -Ramified)	236
6.15	Appendix on Orders (L/F -Unramified)	236
6.16	Appendix on Measures	237
7	A Special Case of the Fundamental Lemma II	239
7.1	The Torus T	239
7.2	The Endoscopic Group M	242
7.3	Orbital Integrals on M	244
7.4	Reduction to H	246
7.5	The Elements of $T(F)$	249
7.6	The Residue Rings $S = o_{E^+}/\pi_F^i$	251
7.7	Integration over $T(F) \backslash H(F)$	256
7.8	Double Cosets in $H(F)$	258
7.9	Summation Conditions for ν	259
7.10	Résumé	261
7.11	The Summation (E/E^+ -Ramified)	262
7.12	The Summation (E/E^+ -Unramified)	263
7.13	Appendix I on Orders (E/E^+ -Ramified)	265
7.14	Appendix II on Orders (E/E^+ -Unramified)	265
7.15	The Global Property	266
8	The Langlands–Shelstad Transfer Factor	271
8.1	An Admissible Homomorphism	271
8.2	Statement of Result	276
8.2.1	a -Data and χ -Data	277
8.2.2	The Element $\omega_T(\sigma)$	278
8.2.3	Lifting of Weyl Group Elements	278
8.2.4	Coroot Modification	279
8.2.5	p -Gauges	281
8.2.6	Weil Groups	281
8.2.7	The 1-Chain r_p for \hat{G}	282
8.2.8	Definition of ξ_χ^G	282
8.2.9	The Similar r_p for \hat{M}	283
8.2.10	Definition of ξ_λ^M	283
8.2.11	The Corestriction Map and the Langlands Reciprocity Law	284
8.2.12	Langlands Reciprocity for the Norm-1-Torus $T = N_L^1$	284
8.2.13	The Definition of Δ_{III_2}	286
8.2.14	The Archimedean Case $F = \mathbb{R}$	288
9	Fundamental Lemma (Twisted Case)	291
9.1	Restriction of Scalars (Notations)	291
9.1.1	Norm Maps	293
9.2	Endoscopy for Twisted Base Change	293
9.3	Matching	296

9.4	The Twisted Fundamental Lemma (Unit Element)	297
9.4.1	The Condition $t \leftrightarrow \delta$	298
9.4.2	The Class $\alpha(t; \delta)$	300
9.4.3	The Frobenius Fundamental Lemma	302
9.4.4	The Support of $O_{\delta\theta}^{G, \kappa}$	303
9.5	The Norm Map and Stable Conjugacy	304
9.6	Comparison of the Transfer Factors Δ and Δ^E	308
9.6.1	The Factor $\Delta_{III}(t, \delta)$	309
9.6.2	The Quotient $\Delta_{III}^E / \Delta_{III}$	311
9.6.3	The Explicit Reciprocity Law	316
9.7	Twisted Transfer Factors for Split Tori	317
9.8	The Case $M_1 \neq M$	319
10	Reduction to Unit Elements	321
10.1	The Unramified Endoscopic Lift	321
10.2	Twisted Transfer Factors for the Split Torus in M_1	323
10.3	The Twisted Spherical Lift	324
10.3.1	The Frobenius Formula	326
10.3.2	The Modified Twisted Lift Π_{mod}	326
10.3.3	Elementary Functions	328
10.3.4	Spherical Projectors	330
10.4	The (Twisted) Fundamental Lemma	330
11	Appendix on Galois Cohomology	335
11.1	Explicit Reciprocity Law for Elementary Characters	336
12	Appendix on Double Cosets	339
12.1	Reduction to Standard Type	342
12.2	The Quadratic Embedding	345
12.3	Elementary Divisors	347
12.4	The Compact Open Groups	349
12.5	The Twisted Group \tilde{H}	351
	References	355
	Index	361
	Nomenclature	365