

Diffusions, Markov Processes, and Martingales

Volume 1: FOUNDATIONS

2nd Edition

L. C. G. ROGERS

*School of Mathematical Sciences,
University of Bath*

and

DAVID WILLIAMS

*Department of Mathematics,
University of Wales, Swansea*



CAMBRIDGE
UNIVERSITY PRESS

Contents

Some Frequently Used Notation	xix
CHAPTER I. BROWNIAN MOTION	
1. INTRODUCTION	1
1. What is Brownian motion, and why study it?	1
2. Brownian motion as a martingale	2
3. Brownian motion as a Gaussian process	3
4. Brownian motion as a Markov process	5
5. Brownian motion as a diffusion (and martingale)	7
2. BASICS ABOUT BROWNIAN MOTION	10
6. Existence and uniqueness of Brownian motion	10
7. Skorokhod embedding	13
8. Donsker's Invariance Principle	16
9. Exponential martingales and first-passage distributions	18
10. Some sample-path properties	19
11. Quadratic variation	21
12. The strong Markov property	21
13. Reflection	25
14. Reflecting Brownian motion and local time	27
15. Kolmogorov's test	31
16. Brownian exponential martingales and the Law of the Iterated Logarithm	31
3. BROWNIAN MOTION IN HIGHER DIMENSIONS	36
17. Some martingales for Brownian motion	36
18. Recurrence and transience in higher dimensions	38
19. Some applications of Brownian motion to complex analysis	39
20. Windings of planar Brownian motion	43
21. Multiple points, cone points, cut points	45

22.	Potential theory of Brownian motion in \mathbb{R}^d ($d \geq 3$)	46
23.	Brownian motion and physical diffusion	51
4.	GAUSSIAN PROCESSES AND LÉVY PROCESSES	55
	<i>Gaussian processes</i>	
24.	Existence results for Gaussian processes	55
25.	Continuity results	59
26.	Isotropic random flows	66
27.	Dynkin's Isomorphism Theorem	71
	<i>Lévy processes</i>	
28.	Lévy processes	73
29.	Fluctuation theory and Wiener–Hopf factorisation	80
30.	Local time of Lévy processes	82
	CHAPTER II. SOME CLASSICAL THEORY	
1.	BASIC MEASURE THEORY	85
	<i>Measurability and measure</i>	
1.	Measurable spaces; σ -algebras; π -systems; d -systems	85
2.	Measurable functions	88
3.	Monotone-Class Theorems	90
4.	Measures; the uniqueness lemma; almost everywhere; a.e. (μ, Σ)	91
5.	Carathéodory's Extension Theorem	93
6.	Inner and outer μ -measures; completion	94
	<i>Integration</i>	
7.	Definition of the integral $\int f d\mu$	95
8.	Convergence theorems	96
9.	The Radon-Nikodým Theorem; absolute continuity; $\lambda \ll \mu$ notation; equivalent measures	98
10.	Inequalities; \mathcal{L}^p and L^p spaces ($p \geq 1$)	99
	<i>Product structures</i>	
11.	Product σ -algebras	101
12.	Product measure; Fubini's Theorem	102
13.	Exercises	104
2.	BASIC PROBABILITY THEORY	108
	<i>Probability and expectation</i>	
14.	Probability triple; almost surely (a.s.); a.s. (\mathbf{P}), a.s. (\mathbf{P}, \mathcal{F})	108

15.	$\limsup E_n$; First Borel–Cantelli Lemma	109
16.	Law of random variable; distribution function; joint law	110
17.	Expectation; $E(X; F)$	110
18.	Inequalities: Markov, Jensen, Schwarz, Tchebychev	111
19.	Modes of convergence of random variables	113
	<i>Uniform integrability and \mathcal{L}^1 convergence</i>	
20.	Uniform integrability	114
21.	\mathcal{L}^1 convergence	115
	<i>Independence</i>	
22.	Independence of σ -algebras and of random variables	116
23.	Existence of families of independent variables	118
24.	Exercises	119
3.	STOCHASTIC PROCESSES	119
	<i>The Daniell–Kolmogorov Theorem</i>	
25.	(E^T, \mathcal{F}^T) ; σ -algebras on function space; cylinders and σ -cylinders	119
26.	Infinite products of probability triples	121
27.	Stochastic process; sample function; law	121
28.	Canonical process	122
29.	Finite-dimensional distributions; sufficiency; compatibility	123
30.	The Daniell–Kolmogorov (DK) Theorem: ‘compact metrizable’ case	124
31.	The Daniell–Kolmogorov (DK) Theorem: general case	126
32.	Gaussian processes; pre-Brownian motion	127
33.	Pre-Poisson set functions	128
	<i>Beyond the DK Theorem</i>	
34.	Limitations of the DK Theorem	128
35.	The role of outer measures	129
36.	Modifications; indistinguishability	130
37.	Direct construction of Poisson measures and subordinators, and of local time from the zero set; Azéma’s martingale	131
38.	Exercises	136
4.	DISCRETE-PARAMETER MARTINGALE THEORY	137
	<i>Conditional expectation</i>	
30.	Fundamental theorem and definition	137
40.	Notation; agreement with elementary usage	138
41.	Properties of conditional expectation: a list	139
42.	The role of versions; regular conditional probabilities and pdfs	140

43.	A counterexample	141
44.	A uniform-integrability property of conditional expectations	142
	<i>(Discrete-parameter) martingales and supermartingales</i>	
45.	Filtration; filtered space; adapted process; natural filtration	143
46.	Martingale; supermartingale; submartingale	144
47.	Previsible process; gambling strategy; a fundamental principle	144
48.	Doob's Upcrossing Lemma	145
49.	Doob's Supermartingale-Convergence Theorem	146
50.	\mathcal{L}^1 convergence and the UI property	147
51.	The Lévy-Doob Downward Theorem	148
52.	Doob's Submartingale and \mathcal{L}^p Inequalities	150
53.	Martingales in \mathcal{L}^2 ; orthogonality of increments	152
54.	Doob decomposition	153
55.	The $\langle M \rangle$ and $[M]$ processes	154
	<i>Stopping times, optional stopping and optional sampling</i>	
56.	Stopping time	155
57.	Optional-stopping theorems	156
58.	The pre- T σ -algebra \mathcal{F}_T	158
59.	Optional sampling	159
60.	Exercises	161
5.	CONTINUOUS-PARAMETER SUPERMARTINGALES	163
	<i>Regularisation: R-supermartingales</i>	
61.	Orientation	163
62.	Some real-variable results	163
63.	Filtrations; supermartingales; R-processes, R-supermartingales	166
64.	Some important examples	167
65.	Doob's Regularity Theorem: Part 1	169
66.	Partial augmentation	171
67.	Usual conditions; R-filtered space; usual augmentation; R-regularisation	172
68.	A necessary pause for thought	174
69.	Convergence theorems for R-supermartingales	175
70.	Inequalities and \mathcal{L}^p convergence for R-submartingales	177
71.	Martingale proof of Wiener's Theorem; canonical Brownian motion	178
72.	Brownian motion relative to a filtered space	180
	<i>Stopping times</i>	
73.	Stopping time T ; pre- T σ -algebra \mathcal{G}_T ; progressive process	181
74.	First-entrance (début) times; hitting times; first-approach times: the easy cases	183

75.	Why 'completion' in the usual conditions has to be introduced	184
76.	Début and Section Theorems	186
77.	Optional Sampling for \mathbb{R} -supermartingales under the usual conditions	188
78.	Two important results for Markov-process theory	191
79.	Exercises	192
6.	PROBABILITY MEASURES ON LUSIN SPACES	200
	' <i>Weak convergence</i> '	
80.	$C(J)$ and $\text{Pr}(J)$ when J is compact Hausdorff	202
81.	$C(J)$ and $\text{Pr}(J)$ when J is compact metrizable	203
82.	Polish and Lusin spaces	205
83.	The $C_b(S)$ topology of $\text{Pr}(S)$ when S is a Lusin space; Prohorov's Theorem	207
84.	Some useful convergence results	211
85.	Tightness in $\text{Pr}(W)$ when W is the path-space $W := C([0, \infty); \mathbb{R})$	213
86.	The Skorokhod representation of $C_b(S)$ convergence on $\text{Pr}(S)$	215
87.	Weak convergence versus convergence of finite-dimensional distributions	216
	' <i>Regular conditional probabilities</i> '	
88.	Some preliminaries	217
89.	The main existence theorem	218
90.	Canonical Brownian Motion $\text{CBM}(\mathbb{R}^N)$; Markov property of \mathbb{P}^x laws	220
91.	Exercises	222
	CHAPTER III. MARKOV PROCESSES	
1.	TRANSITION FUNCTIONS AND RESOLVENTS	227
	1. What is a (continuous-time) Markov process?	227
	2. The finite-state-space Markov chain	228
	3. Transition functions and their resolvents	231
	4. Contraction semigroups on Banach spaces	234
	5. The Hille–Yosida Theorem	237
2.	FELLER–DYNKIN PROCESSES	240
	6. Feller–Dynkin (FD) semigroups	240
	7. The existence theorem: canonical FD processes	243
	8. Strong Markov property: preliminary version	247
	9. Strong Markov property: full version; Blumenthal's 0–1 Law	249

10.	Some fundamental martingales; Dynkin's formula	252
11.	Quasi-left-continuity	255
12.	Characteristic operator	256
13.	Feller–Dynkin diffusions	258
14.	Characterisation of continuous real Lévy processes	261
15.	Consolidation	262
3.	ADDITIVE FUNCTIONALS	263
16.	PCHAFs; λ -excessive functions; Brownian local time	263
17.	Proof of the Volkonskii–Šur–Meyer Theorem	267
18.	Killing	269
19.	The Feynmann–Kac formula	272
20.	A Ciesielski–Taylor Theorem	275
21.	Time-substitution	277
22.	Reflecting Brownian motion	278
23.	The Feller–McKean chain	281
24.	Elastic Brownian motion; the arcsine law	282
4.	APPROACH TO RAY PROCESSES: THE MARTIN BOUNDARY	284
25.	Ray processes and Markov chains	284
26.	Important example: birth process	286
27.	Excessive functions, the Martin kernel and Choquet theory	288
28.	The Martin compactification	292
29.	The Martin representation; Doob–Hunt explanation	295
30.	R. S. Martin's boundary	297
31.	Doob–Hunt theory for Brownian motion	298
32.	Ray processes and right processes	302
5.	RAY PROCESSES	303
33.	Orientation	303
34.	Ray resolvents	304
35.	The Ray–Knight compactification	306
	<i>Ray's Theorem: analytical part</i>	
36.	From semigroup to resolvent	309
37.	Branch-points	313
38.	Choquet representation of 1-excessive probability measures	315
	<i>Ray's Theorem: probabilistic part</i>	
39.	The Ray process associated with a given entrance law	316
40.	Strong Markov property of Ray processes	318
41.	The role of branch-points	319

6. APPLICATIONS	321
<i>Martin boundary theory in retrospect</i>	
42. From discrete to continuous time	321
43. Proof of the Doob–Hunt Convergence Theorem	323
44. The Choquet representation of Π -excessive functions	325
45. Doob h -transforms	327
<i>Time reversal and related topics</i>	
46. Nagasawa's formula for chains	328
47. Strong Markov property under time reversal	330
48. Equilibrium charge	331
49. BM(\mathbb{R}) and BES(3): splitting times	332
<i>A first look at Markov-chain theory</i>	
50. Chains as Ray processes	334
51. Significance of q_i	337
52. Taboo probabilities; first-entrance decomposition	337
53. The Q -matrix; DK conditions	339
54. Local-character condition for Q	340
55. Totally instantaneous Q -matrices	342
56. Last exits	343
57. Excursions from h	345
58. Kingman's solution of the 'Markov characterization problem'	347
59. Symmetrisable chains	348
60. An open problem	349
References for Volumes 1 and 2	351
Index to Volumes 1 and 2	375