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Monographs**

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Geometry and Dynamics in Gromov Hyperbolic Metric Spaces

**With an Emphasis on
Non-Proper Settings**

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