

Contents

<i>Preface</i>	xi
1 Historical Introduction	1
1.1 Euclid and the Parallel Axiom	2
1.2 Spherical and Non-Euclidean Geometry	5
1.3 Vector Geometry	10
1.4 Hilbert's Axioms	14
1.5 Well-ordering and the Axiom of Choice	19
1.6 Logic and Computability	23
2 Classical Arithmetization	26
2.1 From Natural to Rational Numbers	27
2.2 From Rationals to Reals	29
2.3 Completeness Properties of \mathbb{R}	32
2.4 Functions and Sets	35
2.5 Continuous Functions	37
2.6 The Peano Axioms	39
2.7 The Language of PA	43
2.8 Arithmetically Definable Sets	45
2.9 Limits of Arithmetization	48
3 Classical Analysis	51
3.1 Limits	51
3.2 Algebraic Properties of Limits	53
3.3 Continuity and Intermediate Values	55
3.4 The Bolzano-Weierstrass Theorem	57
3.5 The Heine-Borel Theorem	59
3.6 The Extreme Value Theorem	60
3.7 Uniform Continuity	61
3.8 The Cantor Set	64
3.9 Trees in Analysis	66
4 Computability	70
4.1 Computability and Church's Thesis	71

4.2	The Halting Problem	73
4.3	Computably Enumerable Sets	74
4.4	Computable Sequences in Analysis	77
4.5	Computable Tree with No Computable Path	78
4.6	Computability and Incompleteness	80
4.7	Computability and Analysis	81
5	Arithmetization of Computation	85
5.1	Formal Systems	86
5.2	Smullyan's Elementary Formal Systems	87
5.3	Notations for Positive Integers	89
5.4	Turing's Analysis of Computation	91
5.5	Operations on EFS-Generated Sets	93
5.6	Generating Σ_1^0 Sets	96
5.7	EFS for Σ_1^0 Relations	98
5.8	Arithmetizing Elementary Formal Systems	100
5.9	Arithmetizing Computable Enumeration	103
5.10	Arithmetizing Computable Analysis	106
6	Arithmetical Comprehension	109
6.1	The Axiom System ACA_0	110
6.2	Σ_1^0 and Arithmetical Comprehension	111
6.3	Completeness Properties in ACA_0	113
6.4	Arithmetization of Trees	116
6.5	The Kőnig Infinity Lemma	118
6.6	Ramsey Theory	121
6.7	Some Results from Logic	124
6.8	Peano Arithmetic in ACA_0	127
7	Recursive Comprehension	130
7.1	The Axiom System RCA_0	131
7.2	Real Numbers and Continuous Functions	132
7.3	The Intermediate Value Theorem	134
7.4	The Cantor Set Revisited	136
7.5	From Heine-Borel to Weak Kőnig Lemma	137
7.6	From Weak Kőnig Lemma to Heine-Borel	140
7.7	Uniform Continuity	141
7.8	From Weak Kőnig to Extreme Value	143
7.9	Theorems of WKL_0	146
7.10	WKL_0 , ACA_0 , and Beyond	149
8	A Bigger Picture	154
8.1	Constructive Mathematics	155
8.2	Predicate Logic	156
8.3	Varieties of Incompleteness	160

8.4	Computability	162
8.5	Set Theory	164
8.6	Concepts of “Depth”	166
	<i>Bibliography</i>	168
	<i>Index</i>	173