# UNPACKING FRACTIONS

## Classroom-Tested Strategies to Build Students' Mathematical Understanding


**Foreword** by Gail Burrill **xi**

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