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Critical phenomena, quantum field theory, random walks and random surfaces: Some perspectives Part I

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