Contents

Part I: Curricular Perspective	
Preface to Part I	3
Jinfa Cai and Eric Knuth	
Functional Thinking as a Route Into Algebra in the Elementary Grades . Maria L. Blanton and James J. Kaput	5
Introduction	6
The Challenge of Curriculum and Instruction	6
Functional Thinking as a Route to Algebraic Thinking	7
Functional Thinking in the Elementary Grades	8
Children's Capacity for Functional Thinking	9
Integrating Functional Thinking into Curriculum and Instruction Transforming Teachers' Resource Base to Support Students'	16
Functional Thinking	17
Using Children's Functional Thinking to Leverage Teacher Learning	19
Functional Thinking	20
	20
References	21
Developing Students' Algebraic Thinking in Earlier Grades: Lessons	
from China and Singapore	25
Introduction	26
Features of the Chinese and Singaporean Curricula	27
	27
The Chinese Curriculum	28
The Singaporean Curriculum	32



Lessons from Chinese and Singaporean School Mathematics	34
Why Should Curricula Expect Students in Early Grades	
to Think Algebraically?	35
Are Young Children Capable of Thinking Algebraically? How Can We Help Students to Think Arithmetically	36
and Algebraically?	27
Are Authentic Applications Necessary for Students in Early	57
Grades?	38
Conclusion	39
References	40
Developing Algebraic Thinking in the Context of Arithmetic	43
Susan Jo Russell, Deborah Schifter, and Virginia Bastable	
Understanding the Behavior of the Operations	45
Generalizing and Justifying	51
1. Articulating General Claims	51
2. Developing a Mathematical Argument to Justify a General	50
	55
3. Representation-Based Proof: Tools for Proving	
in the Elementary Grades	56
Extending the Number System	59
Using Notation with Meaning	63
Connecting Arithmetic and Algebra	67
References	68
The Role of Theoretical Analysis in Developing Algebraic Thinking:	
A Vygotskian Perspective	71
Jean Schmittau	
Introduction	71
Orienting Children to Theoretical Concepts	74
Role of Psychological Tools	76
The Part-Whole Relation	76
Concluding Remarks	84
References	85
The Arithmetic-Algebra Connection: A Historical-Pedagogical Perspective	87
K. Subramaniam and Kakni Banerjee	07
	8/
Arithmetic and Algebra in the Indian Mathematical Tradition	91
Building on Students' Understanding of Arithmetic	95
The Arithmetic Algebra Connection—A Framework	98
References	105
Shiki: A Critical Foundation for School Algebra in Japanese Elementary	
School Mathematics	109
Tau watallaut	110
School Algeora and Algeora in Early Grades	110

Methodology	111
Algebra in Japanese Curriculum	112
Mathematical Expressions in Japanese Curriculum	114
Mathematical Expressions in Japanese Textbooks	114
Discussion	121
References	123
Commentary on Part I	125
Jeremy Kilpatrick	
Algebra First	126
A Curriculum Topic	127
Numerical Patterns	128
Word Problems	128
Multiple Perspectives	129
References	129
Part II: Cognitive Perspective	
Preface to Part II	135
Eric Knuth and Jinfa Cai	
Algebraic Thinking with and without Algebraic Representation:	
A Pathway for Learning	137
Murray S. Britt and Kathryn C. Irwin	
Introduction	138
Children's Understanding of Generalities for Operations	
Before Schooling	139
Algebraic Thinking and the New Zealand Numeracy Project	140
Students' Algebraic Thinking in the Last Year of Intermediate	
School (Age 11–12)	146
The Growth of Algebraic Thinking from Numbers to Symbols:	
A Longitudinal Study	147
Discussion	152
A Pathway for Algebraic Thinking	153
References	157
Examining Students' Algebraic Thinking in a Curricular Context:	
A Longitudinal Study	161
Jinfa Cai, John C. Moyer, Ning Wang, and Bikai Nie	
Standards-Based and Traditional Curricula in the United States	162
LieCal Project	163
Highlights of the Differences between CMP and Non-CMP	
Curricula	164
Defining Variables	165
Defining Equations	165
Introducing Equation Solving	166
Using Mathematical Problems	168

Highlights of the Differences between CMP and Non-CMP	
Classroom Instruction	. 169
Conceptual and Procedural Emphases	. 170
Instructional Tasks	. 171
Students' Development of Algebraic Thinking: Methodological	
Considerations	. 172
The Focus of Algebraic Thinking	. 173
Tasks and Data Analysis	. 174
Findings about the Development of Students' Algebraic Thinking	. 174
Representing Situations	. 175
Solving Equations	. 177
Making Generalizations	. 178
Conclusions and Instructional Implications	. 180
References	. 183
Years 2 to 6 Students' Ability to Generalise: Models, Representations	
and Theory for Teaching and Learning	. 187
Tom J. Cooper and Elizabeth Warren	
Perspectives on the Mathematics of Early Algebra	. 188
Representation and Generalisation	. 190
Models and Representations	. 191
Generalisation	. 191
Focus of EATP	. 193
Focus of Chapter	. 194
Design of EATP	. 194
Findings and Discussion	. 196
Patterns	. 197
Change and Functions	. 198
Equations and Equivalence	. 201
Generalising Principles and Abstract Representations	. 204
Conclusions and Implications	. 206
Models and Representations	. 206
Generalisation	. 207
Theoretical Framework	. 209
References	. 211
Algebra in the Mildle School: Developing Functional Relationships	215
	. 215
Amy B. Ellis	216
The Importance of (and Diff) him with the state of the st	. 210
Ine Importance of (and Difficulties with) Functional Thinking .	. 218
An Alternative Approach to Function: Quantities and Covariation	. 222
A Flexible Understanding of Functions	. 226
Coordinating Covariation and Correspondence Approaches.	. 226
Flexibility Across Forms	. 230

Fostering a Focus on Quantities	. 234
References	235
Representational Competence and Algebraic Modeling	239
Andrew Izsák	
Early Results on Students' Understandings of Standard	
Representations in Algebra	. 241
Theoretical Accounts of Reasoning with External Representations	. 241
Students' Capacities to Reason with External Representations	. 243
First Result: Criteria for Evaluating External Representations	. 244
Second Result: Adaptive Interpretation	. 249
Conclusion	. 253
References	256
Middle School Students' Understanding of Core Algebraic Concepts:	
Equivalence & Variable	259
Eric J. Knuth, Martha W. Alibali, Nicole M. McNeil, Aaron Weinberg,	
and Ana C. Stephens	
Introduction	260
Student Understanding of Equivalence & Variable	261
Equivalence	. 261
Variable	262
Method	262
Participants	262
Data Collection	263
Coding	264
Results	266
Interpretation of the Equal Sign	266
Performance on the Equivalent Equations Problem	267
Interpretation of a Literal Symbol	270
Performance on the which Is Larger Problem	. 271
Discussion	. 273
Equivalence Results	. 273
Variable Results	274
Concluding Remarks	. 275
References	. 275
An Approach to Geometric and Numeric Patterning that Fosters Second	
Grade Students' Reasoning and Generalizing about Functions	
	217
Joan Moss and Susan London McNab	<u> </u>
	. 277
	. 279
Our Approach: Theoretical	. 279
Instructional Sequence	. 281
Visual Representation: Geometric Growing Patterns	. 281

Numeric Representations: Function Machine	282
Integration Activities: Pattern Sidewalk	283
Role of the Teacher	284
Procedures and Measures: Grade 2 Interventions	285
Results	285
Finding Rules for Patterns and Generating Patterns Based	200
on Given Rules	286
Constructing a Pattern from a Pule: "A 'number times two	200
nlus one' nettern?"	286
Finding a Dula for a Civar Battern, "Desition number times	200
Finding a Rule for a Given Pattern. Position number times	107
	287
Students' Invention of Multiplication	288
Deconstructing Multiplication: "Double the position, plus the position"	289
Using a Structural Understanding of Multiplication to Predict	
Far Positions: "It's 40 up, and 3 to the side"	289
The Discovery of Zero	291
Zero as a Coefficient: "Zero groups of 4 million is zero"	291
Zero as a Position Number: "the zero-th position".	292
Transfer of Structure	293
Circumventing Whole Object Reasoning	293
Informal Algebraic Expressions of Rules in the Sparky	470
Droblem	201
	294
The Curriculum with Its Ecous on Integration	295
Driegitiging Viewal Depresentations of Detterm	290
Prioritizing visual Representations of Pattern	291
Pedagogy and Student Inventions	297
	298
References	298
Grade 2 Students' Non-Symbolic Algebraic Thinking	303
Luis Radford	
Introduction	303
Extending Sequences	305
Abstraction	307
The Boundaries of Arithmetic and Algebraic Thinking	308
Layers of Generality	311
Beyond Intuited Indeterminacy	312
A General Overview	316
Synthesis and Concluding Remarks	317
References	320
Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study F.D. Rivera and Joanne Rossi Becker Anticipating What Is to Come: Initial Reflections on Our Three-Year Data from the Clinical Interviews	323 327

Contents

Cognitive Issues Surrounding Pattern Generalization: What We Know from Various Theoretical Perspectives and Empirical	
Studies	329
Clarifying the Definition of Pattern Generalization	329
Types of Algebraic Generalization Involving Figural Patterns	330
Methodology	331
Classroom Contexts from Years 1 to 3 of the Study	331
Nature and Content of Classroom Teaching Experiments	
in Years 1 and 2	332
Nature and Content of Classroom Teaching Experiments	. ' . ' L
in Year 3	334
Nature and Content of Clinical Interview Tasks from Vears 1	5.54
to 3	335
Date Collection and Analysis and Delevant Study Protocols	225
Findings and Discussion Data 1. Accounting for Construction	555
Findings and Discussion Part 1: Accounting for Constructive	220
and Deconstructive Generalizations	338
Findings and Discussion Part 2: Understanding the Operations	
Needed in Developing a Pattern Generalization	342
Findings and Discussion Part 3: Factors Affecting Students'	
Ability to Develop CGs	344
Findings and Discussion Part 4: A Three-Year Account	
of Classroom Mathematical Practices that Encouraged	
the Formation of Generalization Among Our Middle School	
Students	347
Year 1 Classroom Practices: From Figurally- to	
Numerically-Driven CSGs	348
Year 2 Practice: Continued Use of Numerically-Driven CSGs	
and a Refinement in the Case of Decreasing Linear	
Patterns	351
Year 3 Practices: A Third Shift Back to Figural-based	
Generalization and the Consequent Occurrence	
of CSGs CNGs and DGs	252
Eindings and Discussion Dart 5. Middle School Students'	352
Findings and Discussion Part 5: Middle School Students	251
Capability in Justifying CSOs \dots \dots	. 554
Findings and Discussion Part 6: Middle School Students	
Capability in Constructing and Justifying CNGs and DGs	. 357
	. 362
References	. 363
Commentary on Part II	. 367
Bharath Sriraman and Kyeong-Hwa Lee	
Introductory Remarks	. 367
Early Algebraization Versus Meaningful Arithmetic	. 368
Generalized Arithmetic, Generalizing, Generalization	369
From Haeckel to Lamarck to Early Algebraization	370
References	377

Part III: Instructional Perspective

Preface to Part III	377
Prospective Middle-School Mathematics Teachers' Knowledge	
of Equations and Inequalities	379
Nerida F. Ellerton and M.A. (Ken) Clements	
The Context	379
Mathematical Considerations Relating to the Teaching	
and Learning of Equations and Inequalities	380
Student Misconceptions in Regard to Quadratic Equations	383
Student Misconceptions with Regard to Linear Inequalities	384
The Pre-Service Teachers Involved, and Tasks Used, in the Present	
Study	386
"Clever" Tasks	387
Developing the Pencil-and-Paper Instruments	389
The Eight Equation/Algebraic Inequality Pairs	389
Study Design, and Results	395
Population and Sample Considerations	395
Results	396
Conclusions in Relation to the Prospective Teachers' Knowledge	
of Algebraic Inequalities	399
Prospective Teachers' Knowledge in Relation to Quadratic	
Equations	401
Bad News, Good News and Some Concluding Comments	402
Bad News	402
Good News	403
Student Confidence Considerations	406
Concluding Comments	406
References	407
The Algebraic Nature of Fractions: Developing Relational Thinking	
in Elementary School	409
Susan B. Empson, Linda Levi, and Thomas P. Carpenter	
What Is Relational Thinking?	411
Use of Relational Thinking in Learning Fractions	413
Understanding Fractional Quantities Through Relational	
Thinking	413
Use of Relational Thinking to Make Sense of Operations	
Involving Fractions	416
Discussion of Cases	422
A Conjecture Concerning Relational Thinking as a Tool in	
Learning New Number Content	423

Contents

	425
References	426
Professional Development to Support Students' Algebraic Reasoning:	
An Example from the Problem-Solving Cycle Model	429
Karen Koellner, Jennifer Jacobs, Hilda Borko, Sarah Roberts, and Craig	
Schneider	
Introduction	430
The Problem-Solving Cycle Model of Professional Development.	431
The PSC as Implemented in the STAAR Project	432
Prior Research on the Development and Impact of the PSC	435
Impact of the PSC on Instructional Practice: A Case Study	
	436
Methods	436
Ken Bryant	436
Data Sources	437
Data Analysis	438
Results and Discussion	440
Patterns Drawn from QMI Coding and Analysis	440
Vignette Analysis: Ken's Skyscraper Windows Lesson	447
Conclusions	450
References	451
Using Habermas' Theory of Pationality to Cain Insight into Students'	
Understanding of Algebraic Language	453
Francesca Morselli and Paolo Roero	ч <i>э</i> э
Introduction	453
Habermas' Construct of Rational Behaviour	454
Adaptation of Habermas' Construct of Rational Behavior	
to the Case of the Use of Algebraic Language	455
Epistemic Rationality	455
Teleological Rationality	456
Communicative Rationality	456
Relationships with Other Studies on Proving and Modeling	
and on the Teaching and Learning of Algebra	457
	457
Modeling	459
Teaching and Learning of Algebra	459
Description and Interpretation of Student Behavior	462
Habermas' Analytical Tool: Examples of Analysis of Student	
Behavior at Different School Levels	462
Habermas Analytical Tool: Analysis of a Teaching Experiment	468
The Context of the Study: Description of the Research Project	468
First Task: Choose a Number	469
Second Task: Representing the Game	470
Discussion	477
and on the Teaching and Learning of Algebra Proving Proving Modeling Modeling Teaching and Learning of Algebra Teaching and Learning of Algebra Proving Description and Interpretation of Student Behavior Proving Habermas' Analytical Tool: Examples of Analysis of Student Behavior at Different School Levels Proving Habermas Analytical Tool: Analysis of a Teaching Experiment Proving The Context of the Study: Description of the Research Project Proving First Task: Choose a Number Second Task: Representing the Game Discussion Proving	457 459 459 462 462 468 468 468 469 470 477

Research Advances	477
Educational Implications	478
References	479
Theoretical Issues and Educational Strategies for Encouraging Teachers	
to Promote a Linguistic and Metacognitive Approach to Early	400
Algebra	483
Annalisa Cusi, Nicolina A. Malara, and Giancarlo Navarra	
Introduction	483
In Europe	484
From Traditional Algebra to Early Algebra	485
Early Algebra as a Meta-Subject and the ArAl Project	486
Socio-Constructive Teaching and Teacher Training	487
The Role of the Teacher's Reflection	488
The Role of the ArAl Glossary in Teacher Training	490
Algebraic Babbling	492
Algebraic Babbling \rightarrow Algebra as a Language	493
Algebraic Babbling \rightarrow Syntax, Semantics \rightarrow Brioshi	494
Brioshi \rightarrow Canonical/Non Canonical form of a	
Number \rightarrow '='	495
The Multi-Commented Transcripts Methodology (MCTM)	496
From the Comments to a Classification of Attitudes	499
From the Comments to a Classification of Attitudes	502
Concluding Demorte	502
	507
References	507
A Procedural Focus and a Relationship Focus to Algebra: How U.S.	
Teachers and Japanese Teachers Treat Systems of Equations	511
Margaret Smith	
Background	512
Algebraic Reasoning	512
TIMSS Video Studies	514
	515
	515
	516
	516
Discussion of Key Differences	510
	520
References	520
Teaching Algebraic Equations with Variation in Chinese Classroom	529
ling I i Aibui Peng and Najaing Song	
Introduction	529
The Source of the Data	531
The source of the Data	2.74
	521
The Method of Deserve	531
The Method of Research	531 533 522
The Method of Research	531 533 533

The Improvement of Understanding of Equation	535
Equations Solving	539
The Application of Equations	541
Discussion and Conclusion	545
Process of Teaching Algebra with Variation	545
Operation of Teaching Algebra with Variation	546
Final Comments	548
References	555
Commentary on Part III	557
John Mason	
Introduction	557
Systematics: Structure of Activity	558
What Is Algebra?	559
What Is and What Could Be: Teaching Algebra as an Activity	560
Traditional Algebra Teaching	561
Envisioned Algebra Teaching	563
What Makes 'Algebra' Early?	566
Comparisons	568
Transforming Algebra Teaching and Learning as an Activity	568
How Can Locally Successful Teaching Be Engineered for All?	569
What Is and Could Be Researched?	570
What Is Really Researched?	571
Conclusions	574
References	574
Overall Commentary on Early Algebraization: Perspectives for Research and Teaching	579
Carolyn Kieran Shaaina (ha Nation of Alashasia Thinking within Farly Alashas	FON
Shaping the Notion of Algebraic Thinking within Early Algebra	580
Thinking about the General in the Particular	501
Thinking Rule-Wise about Patterns	582
Thinking Relationally about Quantity, Number, and Numerical	502
	283
Thinking Representationally about the Relations in Problem	
	282
Thinking Conceptually about the Procedural	586
Anticipating, Conjecturing, and Justifying	588
Gesturing, Visualizing, and Languaging	590
The View of Algebraic Thinking that Emerges from this Volume	591
References	592
Author Index	595
Subject Index	609
Editors and Contributors	615