

Contents

Preface	VII
Introduction	1
I Preliminaries	7
1 Lie Groups and Lie Algebras	7
1.1 Lie Groups and an Infinite-Dimensional Setting	7
1.2 The Lie Algebra of a Lie Group	9
1.3 The Exponential Map	12
1.4 Abstract Lie Algebras	15
2 Adjoint and Coadjoint Orbits	17
2.1 The Adjoint Representation	17
2.2 The Coadjoint Representation	19
3 Central Extensions	21
3.1 Lie Algebra Central Extensions	22
3.2 Central Extensions of Lie Groups	24
4 The Euler Equations for Lie Groups	26
4.1 Poisson Structures on Manifolds	26
4.2 Hamiltonian Equations on the Dual of a Lie Algebra	29
4.3 A Riemannian Approach to the Euler Equations	30
4.4 Poisson Pairs and Bi-Hamiltonian Structures	35
4.5 Integrable Systems and the Liouville–Arnold Theorem	38
5 Symplectic Reduction	40
5.1 Hamiltonian Group Actions	41
5.2 Symplectic Quotients	42
6 Bibliographical Notes	44
II Infinite-Dimensional Lie Groups: Their Geometry, Orbits, and Dynamical Systems	47
1 Loop Groups and Affine Lie Algebras	47
1.1 The Central Extension of the Loop Lie algebra	47

1.2	Coadjoint Orbits of Affine Lie Groups	52
1.3	Construction of the Central Extension of the Loop Group	58
1.4	Bibliographical Notes	65
2	Diffeomorphisms of the Circle and the Virasoro–Bott Group	67
2.1	Central Extensions	67
2.2	Coadjoint Orbits of the Group of Circle Diffeomorphisms	70
2.3	The Virasoro Coadjoint Action and Hill’s Operators	72
2.4	The Virasoro–Bott Group and the Korteweg–de Vries Equation	80
2.5	The Bi-Hamiltonian Structure of the KdV Equation	82
2.6	Bibliographical Notes	86
3	Groups of Diffeomorphisms	88
3.1	The Group of Volume-Preserving Diffeomorphisms and Its Coadjoint Representation	88
3.2	The Euler Equation of an Ideal Incompressible Fluid	90
3.3	The Hamiltonian Structure and First Integrals of the Euler Equations for an Incompressible Fluid	91
3.4	Semidirect Products: The Group Setting for an Ideal Magnetohydrodynamics and Compressible Fluids	95
3.5	Symplectic Structure on the Space of Knots and the Landau–Lifschitz Equation	99
3.6	Diffeomorphism Groups as Metric Spaces	105
3.7	Bibliographical Notes	109
4	The Group of Pseudodifferential Symbols	111
4.1	The Lie Algebra of Pseudodifferential Symbols	111
4.2	Outer Derivations and Central Extensions of ψ DS	113
4.3	The Manin Triple of Pseudodifferential Symbols	117
4.4	The Lie Group of α -Pseudodifferential Symbols	119
4.5	The Exponential Map for Pseudodifferential Symbols	122
4.6	Poisson Structures on the Group of α -Pseudodifferential Symbols	124
4.7	Integrable Hierarchies on the Poisson Lie Group \tilde{G}_{INT}	129
4.8	Bibliographical Notes	132
5	Double Loop and Elliptic Lie Groups	134
5.1	Central Extensions of Double Loop Groups and Their Lie Algebras	134
5.2	Coadjoint Orbits	136
5.3	Holomorphic Loop Groups and Monodromy	138
5.4	Digression: Definition of the Calogero–Moser Systems	142
5.5	The Trigonometric Calogero–Moser System and Affine Lie Algebras	146
5.6	The Elliptic Calogero–Moser System and Elliptic Lie Algebras	149
5.7	Bibliographical Notes	152

III Applications of Groups: Topological and Holomorphic Gauge Theories	155
1 Holomorphic Bundles and Hitchin Systems	155
1.1 Basics on Holomorphic Bundles	155
1.2 Hitchin Systems	159
1.3 Bibliographical Notes	162
2 Poisson Structures on Moduli Spaces	163
2.1 Moduli Spaces of Flat Connections on Riemann Surfaces	163
2.2 Poincaré Residue and the Cauchy–Stokes Formula	170
2.3 Moduli Spaces of Holomorphic Bundles	173
2.4 Bibliographical Notes	179
3 Around the Chern–Simons Functional	180
3.1 A Reminder on the Lagrangian Formalism	180
3.2 The Topological Chern–Simons Action Functional	184
3.3 The Holomorphic Chern–Simons Action Functional	187
3.4 A Reminder on Linking Numbers	189
3.5 The Abelian Chern–Simons Path Integral and Linking Numbers	192
3.6 Bibliographical Notes	196
4 Polar Homology	197
4.1 Introduction to Polar Homology	197
4.2 Polar Homology of Projective Varieties	202
4.3 Polar Intersections and Linkings	206
4.4 Polar Homology for Affine Curves	209
4.5 Bibliographical Notes	211
Appendices	213
A.1 Root Systems	213
1.1 Finite Root Systems	213
1.2 Semisimple Complex Lie Algebras	215
1.3 Affine and Elliptic Root Systems	216
1.4 Root Systems and Calogero–Moser Hamiltonians	218
A.2 Compact Lie Groups	221
2.1 The Structure of Compact Groups	221
2.2 A Cohomology Generator for a Simple Compact Group	224
A.3 Krichever–Novikov Algebras	225
3.1 Holomorphic Vector Fields on \mathbb{C}^* and the Virasoro Algebra	225
3.2 Definition of the Krichever–Novikov Algebras and Almost Grading	226
3.3 Central Extensions	228
3.4 Affine Krichever–Novikov Algebras, Coadjoint Orbits, and Holomorphic Bundles	231
A.4 Kähler Structures on the Virasoro and Loop Group Coadjoint Orbits	234

4.1	The Kähler Geometry of the Homogeneous Space Diff(S^1)/ S^1	234
4.2	The Action of Diff(S^1) and Kähler Geometry on the Based Loop Spaces	237
A.5	Diffeomorphism Groups and Optimal Mass Transport	240
5.1	The Inviscid Burgers Equation as a Geodesic Equation on the Diffeomorphism Group	240
5.2	Metric on the Space of Densities and the Otto Calculus	244
5.3	The Hamiltonian Framework of the Riemannian Submersion	247
A.6	Metrics and Diameters of the Group of Hamiltonian Diffeomorphisms	250
6.1	The Hofer Metric and Bi-invariant Pseudometrics on the Group of Hamiltonian Diffeomorphisms	250
6.2	The Infinite L^2 -Diameter of the Group of Hamiltonian Diffeomorphisms	252
A.7	Semidirect Extensions of the Diffeomorphism Group and Gas Dynamics	256
A.8	The Drinfeld–Sokolov Reduction	260
8.1	The Drinfeld–Sokolov Construction	260
8.2	The Kupershmidt–Wilson Theorem and the Proofs	263
A.9	The Lie Algebra \mathfrak{gl}_∞	267
9.1	The Lie Algebra \mathfrak{gl}_∞ and Its Subalgebras	267
9.2	The Central Extension of \mathfrak{gl}_∞	268
9.3	q -Difference Operators and \mathfrak{gl}_∞	269
A.10	Torus Actions on the Moduli Space of Flat Connections	272
10.1	Commuting Functions on the Moduli Space	272
10.2	The Case of SU(2)	274
10.3	SL(n, \mathbb{C}) and the Rational Ruijsenaars–Schneider System	277
References		281
Index		301