

Thomas Runst  
Winfried Sickel

**Sobolev Spaces of Fractional Order,  
Nemytskij Operators,  
and Nonlinear  
Partial Differential Equations**



Walter de Gruyter · Berlin · New York 1996

# Contents

<b>1</b>	<b>Introduction</b>	1
<b>2</b>	<b>Function spaces of Besov–Triebel–Lizorkin type</b>	5
2.1	Definitions and fundamental properties . . . . .	5
2.1.1	Definitions . . . . .	5
2.1.2	Classical function spaces and their appearance in the scales $F_{p,q}^s$ and $B_{p,q}^s$ . . . . .	11
2.1.3	Basic properties . . . . .	14
2.1.4	Lifting property and related quasi-norms . . . . .	18
2.1.5	Dual spaces . . . . .	19
2.1.6	Supplements: Fourier multipliers and maximal inequalities .	21
2.2	Embeddings . . . . .	29
2.2.1	Elementary embeddings . . . . .	29
2.2.2	Embeddings with constant smoothness . . . . .	30
2.2.3	Embeddings with constant differential dimension . . . . .	31
2.2.4	Embeddings in $L_\infty$ , $L_1^{\text{loc}}$ and $L_p$ . . . . .	32
2.2.5	Embeddings for spaces of bounded functions . . . . .	37
2.3	Some equivalent characterizations of $F_{p,q}^s$ and $B_{p,q}^s$ . . . . .	41
2.3.1	Characterizations by differences and some representatives of $F_{p,q}^s$ and $B_{p,q}^s$ . . . . .	41
2.3.2	Nikol'skij representations . . . . .	58
2.3.3	Wavelets and atoms . . . . .	62
2.3.4	A localization property and Fubini-type theorems . . . . .	69
2.4	Spaces on domains . . . . .	72
2.4.1	Preliminaries and definition . . . . .	72
2.4.2	Traces and extensions . . . . .	75
2.4.3	An appropriate lifting operator . . . . .	76
2.4.4	Embeddings, density and duality . . . . .	81
2.4.5	Intrinsic characterizations . . . . .	84
2.5	Interpolation . . . . .	85
2.5.1	The real method . . . . .	86
2.5.2	The complex method . . . . .	86
2.5.3	The $\pm$ -method of Gustafsson–Peetre . . . . .	87
2.5.4	Interpolation of nonlinear operators . . . . .	87
2.6	Homogeneous spaces and a further supplement . . . . .	92
2.6.1	Definition . . . . .	93

2.6.2	Some basic properties . . . . .	94
2.6.3	The spaces $L_{p(r)}^A(\ell_q^s)$ and $\ell_p^s(L_{p(r)}^A)$ . . . . .	95
<b>3</b>	<b>Regular elliptic boundary value problems</b>	<b>99</b>
3.1	Definitions and preliminaries . . . . .	99
3.1.1	Introduction . . . . .	99
3.1.2	Definitions . . . . .	100
3.2	Estimates of integral operators . . . . .	102
3.2.1	A class of integral operators and its symbols . . . . .	102
3.2.2	Estimates of $\mathbf{K}$ in $\dot{F}_{p,q}^s$ and $\dot{B}_{p,q}^s$ . . . . .	105
3.2.3	Estimates of $\mathbf{K}$ . Non-homogeneous spaces . . . . .	110
3.3	Estimates of the Poisson integral . . . . .	113
3.4	A priori estimates . . . . .	117
3.4.1	Liouville theorems . . . . .	117
3.4.2	A priori estimates. Part 1: $\mathbb{R}_+^n$ , constant coefficients . . . . .	119
3.4.3	A priori estimates. Part 2: bounded $C^\infty$ -domains, variable coefficients . . . . .	121
3.5	Regular elliptic boundary value problems . . . . .	124
3.5.1	Dual rich quasi-Banach spaces, Fredholm maps . . . . .	124
3.5.2	Homogeneous boundary conditions . . . . .	130
3.5.3	Non-homogeneous boundary conditions . . . . .	135
3.5.4	Maximum principle . . . . .	137
3.5.5	Counterexamples . . . . .	139
<b>4</b>	<b>Pointwise multiplication</b>	<b>142</b>
4.1	Introduction . . . . .	142
4.2	The definition of the product . . . . .	143
4.2.1	The definition of the product in $\mathcal{S}'$ . . . . .	143
4.2.2	The definition of the product in $D'(\Omega)$ . . . . .	149
4.3	Necessary conditions for pointwise multiplication . . . . .	150
4.3.1	Necessary conditions in the general case . . . . .	150
4.3.2	Necessary conditions in case $m = 2$ . . . . .	160
4.4	Products of a function and a distribution . . . . .	163
4.4.1	A decomposition principle: paraproducts . . . . .	163
4.4.2	Preliminaries. Basic estimates of the paraproducts . . . . .	165
4.4.3	Products of a distribution and a function. Part I . . . . .	171
4.4.4	Products of a distribution and a function. Part II . . . . .	176
4.5	Products of functions and a distribution. The general case . . . . .	181
4.5.1	Products in spaces with negative smoothness . . . . .	181
4.5.2	Products in spaces of positive smoothness . . . . .	187
4.6	The case of constant $p$ . . . . .	190
4.6.1	General results . . . . .	190
4.6.2	Products with a bounded factor . . . . .	197

4.6.3	Characteristic functions as multipliers . . . . .	207
4.6.4	Multiplication algebras . . . . .	221
4.7	The extremal case $p_1 = p$ and $p_2 = \infty$ . . . . .	228
4.7.1	Multiplication with $B_{\infty,q}^s$ . . . . .	229
4.7.2	Multiplication with $F_{\infty,q}^s$ . . . . .	230
4.8	Generalized Hölder inequalities . . . . .	232
4.8.1	Necessary conditions . . . . .	232
4.8.2	Hölder inequalities in case $s > 0$ . . . . .	238
4.8.3	Hölder inequalities in case $s = 0$ . . . . .	240
4.9	The spaces $A_{p,q,\text{unif}}^s$ and relations to $M(A_{p,q}^s)$ . . . . .	246
4.9.1	Embeddings for $M(F_{p,q}^s)$ . . . . .	247
4.9.2	Embeddings for $M(B_{p,q}^s)$ . . . . .	252
4.9.3	A final remark to the definition of the product . . . . .	256
<b>5</b>	<b>Nemytskij operators in spaces of Besov–Triebel–Lizorkin type</b> . . . . .	<b>260</b>
5.1	Introduction . . . . .	260
5.2	Nemytskij operators in Lebesgue and Sobolev spaces . . . . .	261
5.2.1	Some preliminaries . . . . .	261
5.2.2	Nemytskij operators in Lebesgue spaces . . . . .	264
5.2.3	Nemytskij operators in Sobolev spaces $W_p^1(\Omega)$ . . . . .	266
5.2.4	Composition operators on Sobolev spaces $W_p^m$ . . . . .	267
5.2.5	Composition operators on subspaces of $W_p^m$ . . . . .	278
5.3	The Composition operator corresponding to a $C^\infty$ -function $G$ in $F_{p,q}^s$ and $B_{p,q}^s$ . . . . .	290
5.3.1	Necessary conditions . . . . .	290
5.3.2	Powers of $f$ . . . . .	312
5.3.3	Composition operators generated by smooth unbounded $G$ . Part I. Preliminaries . . . . .	316
5.3.4	Composition operators generated by smooth unbounded $G$ . Part II. Bounded functions . . . . .	323
5.3.5	Composition operators generated by smooth unbounded $G$ . Part III. Unbounded functions . . . . .	326
5.3.6	Composition operators corresponding to $G \in C^\infty(\mathbb{R})$ . . . . .	334
5.3.7	Composition operators on $F_{p,q}^s \cap F_{ps,v}^1$ and on $F_{p,q}^s \cap L_\infty$ . . . . .	344
5.4	Powers of $f$ . . . . .	350
5.4.1	The regularity of the absolute value of $f$ . . . . .	350
5.4.2	Sublinear functions $G$ . . . . .	360
5.4.3	Fractional powers $ f ^\mu$ , $\mu > 1$ . . . . .	363
5.4.4	Fractional powers $ f ^\mu$ , $\mu < 1$ . . . . .	365
5.5	Supplements . . . . .	367
5.5.1	$\mathbb{R}^m \rightarrow \mathbb{R}$ -functions $G$ . . . . .	367
5.5.2	Continuity of composition operators . . . . .	372

5.5.3	Differentiability of composition operators . . . . .	378
5.5.4	Nemytskij operators and pseudodifferential operators . . . .	383
<b>6</b>	<b>Applications to semilinear elliptic boundary problems</b>	<b>393</b>
6.1	Introduction . . . . .	393
6.2	The admissibility of spaces of Besov–Triebel–Lizorkin type . . . .	397
6.2.1	The Brouwer degree of a map . . . . .	397
6.2.2	The Leray–Schauder degree . . . . .	398
6.2.3	Topological degree in $B_{p,q}^s$ and $F_{p,q}^s$ . . . . .	400
6.3	Nonlinear perturbations of linear invertible operators . . . . .	412
6.3.1	An abstract result . . . . .	412
6.3.2	Bounded nonlinearities . . . . .	414
6.3.3	Sublinear nonlinearities . . . . .	416
6.3.4	Nonlinearities with linear growth . . . . .	418
6.3.5	Superlinear nonlinearities . . . . .	419
6.4	Results of Landesman–Lazer type . . . . .	423
6.4.1	The Ljapunov–Schmidt method . . . . .	424
6.4.2	The alternative lemma . . . . .	425
6.4.3	Bounded nonlinearities . . . . .	428
6.4.4	Sublinear nonlinearities . . . . .	436
6.4.5	Boundary value problems whose nonlinearities are of linear growth . . . . .	439
6.5	Results of Kazdan–Warner type . . . . .	449
6.5.1	Abstract results . . . . .	450
6.5.2	Applications to semilinear elliptic boundary value problems	454
6.5.3	Solvability of equations depending on a parameter . . . . .	460
6.6	Results of Ambrosetti–Prodi type . . . . .	467
6.6.1	Inversion problems in quasi-Banach spaces . . . . .	469
6.6.2	Singularity theory in quasi-Banach spaces . . . . .	473
6.6.3	Applications to nonlinear elliptic boundary value problems	487
6.6.4	Further multiplicity results . . . . .	513
	Bibliography	525
	Index	545