

Hilbert Transforms in Signal Processing

Stefan L. Hahn

Artech House
Boston • London

Contents

| | |
|--|------|
| Preface | xiii |
| Introduction | 1 |
| Chapter 1 Theory of the One-Dimensional Hilbert Transformation | 3 |
| 1.1 The Concepts of the Hilbert and Fourier Transformations | 3 |
| 1.2 Analytic Functions | 5 |
| 1.3 Cauchy Integral Representation of the Analytic Function: The Analytic Signal | 7 |
| 1.4 Examples of Derivation of Hilbert Transforms in the Time Domain | 9 |
| 1.4.1 Harmonic Signals: $\cos(\omega t)$ and $\sin(\omega t)$, $\omega = 2\pi f$ | 9 |
| 1.4.2 The Hilbert Transform of the Delta Pulse | 10 |
| 1.4.3 The Hilbert Transform of a Square Pulse | 11 |
| 1.5 The Fourier Transform of the Hilbert Transform | 12 |
| 1.6 Symmetry Properties of the Hilbert Transform | 13 |
| 1.7 The Derivation of Hilbert Transforms by Means of Fourier Transforms | 14 |
| 1.7.1 The Hilbert Transform of a Gaussian Pulse | 15 |
| 1.8 The Derivation of Hilbert Transforms Using Hartley Transforms | 16 |
| 1.9 Hilbert Transforms of Periodic Signals | 19 |
| 1.9.1 The Method Based on the Woodward Definition of a Periodic Signal | 19 |
| 1.9.2 The Cotangent Form of the Hilbert Transform of the Periodic Signal | 21 |
| 1.9.3 The Hilbert Transform of the Fourier Series Expansion of a Periodic Function | 22 |

| | | |
|-----------|--|----|
| 1.9.4 | The Derivation of the Hilbert Transform of Periodic Signals Directly From the Time-Domain Hilbert Integral | 25 |
| 1.10 | Hilbert Transforms of the Bessel Functions of the First Kind | 27 |
| 1.10.1 | Derivation of the Hilbert Transforms of Bessel Functions Using Fourier Transforms | 30 |
| 1.11 | One-Sided Spectra of Analytic Signals and Analytic Spectra of Causal Signals | 36 |
| 1.11.1 | One-Sided Spectra of Analytic Signals | 37 |
| 1.11.2 | Analytic Spectra of One-Sided (Causal) Signals | 39 |
| 1.12 | Integration of Analytic Signals | 42 |
| 1.13 | The Definitions of the Instantaneous Amplitude, Phase, and Frequency of Analytic Signals | 43 |
| 1.13.1 | Polar Notation of Analytic Signals | 44 |
| 1.13.2 | The Instantaneous Complex Phase and Complex Frequency | 47 |
| 1.14 | Negative Instantaneous Frequency of the Analytic Signal | 51 |
| | References | 53 |
| Chapter 2 | Properties of the Hilbert Transformation Derivations and Applications | 55 |
| 2.1 | Introduction | 55 |
| 2.2 | Linearity | 55 |
| 2.3 | Linearity: A Method of Generating the Hilbert Transform Using the Samples of a Function | 58 |
| 2.4 | Linearity: Hilbert Transforms of Hyperbolic Functions | 59 |
| 2.5 | Iteration | 60 |
| 2.6 | Differentiation | 61 |
| 2.7 | Successive Iteration and Differentiation | 62 |
| 2.8 | Differentiation of the Convolutions | 65 |
| 2.9 | Differentiation and Multiplication by t . Hilbert Transforms of Hermite Polynomials and Functions | 67 |
| 2.9.1 | Hermite Polynomials | 68 |
| 2.10 | Hilbert Transforms of Legendre Polynomials | 74 |
| 2.10.1 | Hilbert Transforms of Legendre Polynomials by Fourier Transforms | 80 |
| 2.11 | Autoconvolution, Autocorrelation, and Energy Equality | 82 |
| 2.12 | The n -Fold Autoconvolution | 86 |
| 2.13 | The Hilbert Transform of a Product of Two Signals | 88 |
| 2.13.1 | Nonoverlapping Spectra of $f(t)$ and $g(t)$ - Bedrosian's Theorem | 88 |
| 2.13.2 | The Hilbert Transform of the Product $a(t)\cos(\omega_0 t + \varphi_0)$ [5] | 91 |
| 2.14 | The Hilbert Transform of a Product of Analytic Signals | 92 |
| | References | 93 |
| Chapter 3 | Distributions in the Theory of the Hilbert Transformation and Complex Signals | 95 |
| 3.1 | Introduction | 95 |

| | | |
|-----------|---|-----|
| 3.2 | Definition of a Distribution in Terms of Functionals | 96 |
| 3.3 | The Complex Delta Distribution | 97 |
| 3.3.1 | A Convolution Definition of the Analytic Signal | 99 |
| 3.3.2 | The Concept of the Inverse Distribution | 100 |
| 3.4 | The Polar Notation of the Complex Delta Distribution | 101 |
| 3.5 | Other Notations of the Complex Delta Distribution | 104 |
| 3.6 | The Integral and the Derivatives of the Complex Delta Distribution in Terms of the Cauchy Approximation Functions | 104 |
| 3.7 | The Integral and the Derivatives of the Complex Delta Distribution | 106 |
| 3.8 | The Complex Delta Sampling Sequence | 107 |
| 3.9 | The Two-Dimensional Complex Delta Distribution | 110 |
| 3.10 | The Polar Representation of the Two-Dimensional Complex Delta Distribution | 112 |
| 3.10.1 | The Illustration of the Two-Dimensional Complex Delta Distribution With Approximation Functions | 114 |
| 3.11 | The Two-Dimensional Complex Sampling Sequence | 115 |
| 3.12 | The Three-Dimensional Complex Delta Distribution | 116 |
| | References | 118 |
| Chapter 4 | The Discrete Hilbert Transformation | 121 |
| 4.1 | Introduction | 121 |
| 4.2 | The DFT—Discrete Fourier Transformation | 121 |
| 4.2.1 | The Illustration of the Evenness and Oddness of Sequences | 124 |
| 4.2.2 | The dc and ac Parts of a Sequence | 125 |
| 4.3 | Examples of the Derivation of the DFT for Selected Simple Signals | 127 |
| 4.3.1 | The DFT of Trigonometric Functions | 134 |
| 4.4 | The Z-Transformation | 136 |
| 4.5 | The Elementary Properties of the DFT and Z-Transformations | 138 |
| 4.5.1 | Linearity | 138 |
| 4.5.2 | Energy Equality (Parseval's Theorem) | 139 |
| 4.5.3 | Circular Convolution: Convolution-to-Multiplication Theorem | 140 |
| 4.5.4 | Shifting Property | 141 |
| 4.6 | The Discrete Hilbert Transformation | 141 |
| 4.6.1 | Energy Relations | 145 |
| 4.7 | Discrete Hilbert Transforms of Selected Sequences | 146 |
| 4.7.1 | Energy Relations | 146 |
| 4.7.2 | The Hilbert Transform of the Kronecker Delta Sample | 147 |
| 4.7.3 | The Hilbert Transforms of Trigonometric Functions | 149 |
| 4.7.4 | Energy Relations | 149 |
| 4.7.5 | The Hilbert Transform of a Gaussian Sequence | 149 |
| 4.7.6 | The DHT of a Sampled Unipolar Square Pulse | 150 |
| 4.8 | Iteration of the Discrete Hilbert Transformation | 153 |
| 4.8.1 | Energy Relation | 155 |

| | | |
|---------------------------------------|---|-----|
| 4.9 | The System Theory Derivation of the DHT | 155 |
| 4.10 | The Complex Analytic Discrete Sequence | 156 |
| 4.11 | Causal Discrete-Time Sequences and Analytic Discrete Spectra | 159 |
| 4.12 | The Bilinear Transformation and the Cotangent Form of Hilbert Transformations | 162 |
| | References | 167 |
| Chapter 5 Hilbert Transformers | | |
| 5.1 | General Features of Hilbert Transformers | 169 |
| 5.1.1 | Transfer Function and Bandwidth | 169 |
| 5.2 | Phase-Splitter Hilbert Transformers | 171 |
| 5.2.1 | Analog All-Pass Filters | 172 |
| 5.3 | A Simple Method of Design for Hilbert Phase Splitters | 174 |
| 5.3.1 | First Step | 175 |
| 5.3.2 | Second Step | 176 |
| 5.3.3 | Delay, Phase Distortions, and Equalization | 183 |
| 5.4 | Hilbert Transformers With Tapped Delay Line Filters | 184 |
| 5.5 | Bandpass Hilbert Transformers | 186 |
| 5.6 | Generation of Hilbert Transforms Using SSB Filtering | 192 |
| 5.7 | Digital Hilbert Transformers | 193 |
| 5.7.1 | The Transfer Function of the Ideal Noncausal Hilbert Transformer | 194 |
| 5.7.2 | Types of Digital Hilbert Transformers | 198 |
| 5.8 | FIR Hilbert Transformers [15–18] | 198 |
| 5.8.1 | Design | 198 |
| 5.8.2 | Rectangular Window | 200 |
| 5.8.3 | Parameters of the $G(e^{j\psi})$ Function | 202 |
| 5.8.4 | Improving the Ripple Distribution Using Windows | 203 |
| 5.8.5 | Types of Windows | 203 |
| 5.8.6 | Illustration of The Functions $G(e^{j\psi})$ Obtained by Using Various Windows | 206 |
| 5.8.7 | Comparison of the Parameters of FIR Hilbert Transformers Designed Using the Rectangular, Kaiser, and Tschebysheff Windows | 208 |
| 5.8.8 | Kaiser Window With Preemphasis | 210 |
| 5.8.9 | Design of FIR Hilbert Transformers With Even Values of N | 212 |
| 5.8.10 | Derivation of the Transfer Function (5.48) | 212 |
| 5.9 | FIR Hilbert Transformers With Halfband Filters | 215 |
| 5.10 | Recapitulation of the Design Procedure of FIR Hilbert Transformers | 219 |
| 5.11 | Digital All-Pass Hilbert Transformers [23] | 222 |
| 5.12 | The Design of Hilbert Transformers by Use of Bilinear Frequency Transformation | 225 |

| | | |
|---|---|-----|
| 5.13 | IIR Discrete-Time Hilbert Transformers | 226 |
| 5.13.1 | Butterworth Phase Functions | 229 |
| 5.14 | Differentiating Hilbert Transformers | 230 |
| 5.14.1 | Analog Relations | 230 |
| 5.14.2 | Discrete-Time Relations | 230 |
| 5.14.3 | The Design of the FIR Differentiating Hilbert Transformer | 233 |
| | References | 239 |
| Chapter 6 The Hilbert Transform In Modulation Theory | | 241 |
| 6.1 | Introduction | 241 |
| 6.1.1 | Definition | 241 |
| 6.2 | The Concept of the Modulation Function of a Harmonic Carrier | 242 |
| 6.2.1 | The Modified Modulation Function | 243 |
| 6.3 | Classification | 244 |
| 6.3.1 | Linear Modulation | 245 |
| 6.3.2 | Nonlinear Modulation | 245 |
| 6.4 | Test Signals in Modulation | 247 |
| 6.4.1 | Test Signal in the Form of a Fourier Series | 247 |
| 6.4.2 | Test Signals With Random Phases of the Harmonic Terms of the Fourier Series | 252 |
| 6.5 | Basic Theory of Amplitude Modulation | 253 |
| 6.5.1 | AM Modulators | 253 |
| 6.5.2 | Low-Pass-to-Bandpass Filtering Analogy in AM | 256 |
| 6.5.3 | AM—Energy Relations | 258 |
| 6.6 | Basic Theory of Angle Modulation | 260 |
| 6.6.1 | Classification | 262 |
| 6.6.2 | Phase and Frequency Modulators | 263 |
| 6.6.3 | Spectra of Angle Modulation: Harmonic Modulating Signal | 263 |
| 6.6.4 | Frequency Modulation by the Harmonic Signal | 266 |
| 6.6.5 | Narrowband Phase or Frequency Modulation | 266 |
| 6.6.6 | Wideband Phase or Frequency Modulation | 267 |
| 6.6.7 | Adiabatic Theorem | 267 |
| 6.6.8 | Spectra of Angle Modulation: Multitone Modulating Signal | 267 |
| 6.7 | Single-Sideband Linear AM Modulation | 269 |
| 6.7.1 | Single-Sideband Modulators | 271 |
| 6.8 | General Forms of Single-Sideband Modulations | 272 |
| 6.8.1 | Basic Relations | 272 |
| 6.9 | The CSSB Signal for a Linear Envelope Demodulator | 274 |
| 6.10 | CSSB Modulation for Square-Law AM Demodulator | 277 |
| 6.11 | SSB Signal for a Linear FM Demodulator | 280 |
| 6.11.1 | Applications of the CSSB Signals | 281 |
| | References | 283 |

| | | |
|-----------|---|-----|
| Chapter 7 | The Hilbert Transform in Signal and System Theory | 285 |
| 7.1 | Introduction | 285 |
| 7.2 | Hilbert Transforms in the Theory of Linear Systems: Kramers-Kronig Relations | 285 |
| 7.2.1 | Causality | 286 |
| 7.2.2 | Physical Realizability of Transfer Functions | 287 |
| 7.2.3 | Minimum Phase Property | 288 |
| 7.3 | Amplitude Phase Relations in DLTI Systems | 290 |
| 7.3.1 | Minimum Phase Property in DLTI Systems | 292 |
| 7.4 | Measurement Systems Using the Amplitude Phase Relations of LTI Systems | 293 |
| 7.5 | The Kramers-Kronig Relations in Linear Macroscopic Continuous Media | 293 |
| 7.6 | The Concept of Signal Delay in the Hilbertian Sense | 295 |
| 7.7 | The Hilbert Transform in the Theory of Sampling | 298 |
| 7.7.1 | Bandpass Filtering of the Low-Pass Sampled Signal | 303 |
| 7.8 | Sampling of Bandpass Signals | 304 |
| 7.9 | Quadrature Sampling of a Bandpass Signal | 305 |
| 7.10 | Linear Transformations of the Interpolatory Expansion | 305 |
| 7.11 | Generation of a Random Signal Using the Interpolation Expansion | 306 |
| 7.12 | Complex Analytic Random Signals | 308 |
| 7.12.1 | Spectral Moments | 313 |
| 7.12.2 | The Instantaneous Phase and Frequency of Gaussian Low-Pass Noise | 314 |
| 7.13 | Instantaneous Spectral Moments Defined by the Wiegner-Ville Time-Frequency Distribution | 316 |
| 7.14 | Generation of Two-Dimensional Random Fields | 319 |
| 7.14.1 | Generation of Time-Variable Two-Dimensional Random Fields | 319 |
| 7.15 | The Definition of Electrical Power in Terms of Hilbert Transforms and Analytic Signals | 321 |
| 7.15.1 | Harmonic Waveforms of Voltage and Current | 321 |
| 7.15.2 | The Notion of Complex Power | 324 |
| 7.15.3 | Generalization of the Notion of Power | 324 |
| 7.16 | Generalization of the Notion of Power for Signals With Finite Average Power | 326 |
| | References | 331 |
| Chapter 8 | Multidimensional Complex Signals and Applications | 333 |
| 8.1 | Introduction | 333 |
| 8.2 | The Definition of the Multidimensional Complex Signal | 333 |
| 8.2.1 | The Frequency-Domain Definition of the n -Dimensional Complex Signal | 334 |

| | | |
|------------|---|-----|
| 8.2.2 | Signal Domain Definition of the n -Dimensional Complex Signal | 335 |
| 8.3 | The Definition of the Two-Dimensional Complex Signal | 336 |
| 8.3.1 | The Kernel of the Two-Dimensional Fourier Transformation Written in Terms of Hilbert Transforms | 337 |
| 8.4 | Separable Two-Dimensional Signals | 339 |
| 8.5 | Conjugate Two-Dimensional Analytic Signals | 340 |
| 8.6 | The Polar Notation of Two-Dimensional Complex Signals | 341 |
| 8.6.1 | Local Amplitudes and Phases for Separable Two-Dimensional Signals | 343 |
| 8.7 | Examples of Separable Complex Two-Dimensional Signals | 344 |
| 8.8 | Examples of Nonseparable Two-Dimensional Complex Signals | 348 |
| 8.9 | Image Decomposition and Reconstruction Using Amplitude and Phase Patterns | 358 |
| 8.10 | Three-Dimensional Complex Signals | 360 |
| 8.11 | Multidimensional Modulation Theory | 363 |
| 8.11.1 | The n -Dimensional Modulation Function | 364 |
| 8.11.2 | The Two-Dimensional Modulation Theory | 364 |
| 8.11.3 | Two-Dimensional Amplitude Modulation | 365 |
| 8.11.4 | The Single-Quadrant Modulation SQM | 366 |
| 8.11.5 | Two-Dimensional Phase Modulation | 367 |
| | References | 368 |
| Chapter 9 | Multidimensional Hilbert and Fourier Transformations | 369 |
| 9.1 | Introduction | 369 |
| 9.2 | Evenness and Oddness of Multidimensional Real Signals | 369 |
| 9.3 | Signal Domain Definition of the n -Dimensional Hilbert Transformation | 373 |
| 9.4 | Two-Dimensional Hilbert Transformations | 374 |
| 9.5 | Partial Hilbert Transformations | 375 |
| 9.6 | The Derivation of n -Dimensional Hilbert Transforms by Means of n -Dimensional Fourier Transforms | 376 |
| 9.7 | The Two-Dimensional Discrete Hilbert Transformation: Two-Dimensional DHT | 379 |
| 9.7.1 | Properties of the One-Dimensional DHT | 380 |
| 9.7.2 | The Two-Dimensional Discrete Hilbert Transform | 380 |
| 9.7.3 | Properties of the Two-Dimensional DHT [4] | 381 |
| 9.7.4 | Energy Relations | 385 |
| 9.8 | Stark's Extension of Bedrosian's Theorem | 391 |
| 9.9 | Tables of Two-Dimensional Hilbert Transforms | 394 |
| | References | 395 |
| Appendix A | Tabulation of Hilbert Pairs | 397 |

| | | |
|------------------|---|-----|
| Appendix B | The Derivation of the Derivative of the Logarithmic and $\theta(t)$ Distributions | 405 |
| Appendix C | Supplement to Chapter 4 | 407 |
| C.1 | The Fourier Series Representation of the DFT | 407 |
| C.2 | Calculation of the DHT Using the Discrete Hartley Transformation | 408 |
| Appendix D | Details of the Calculation of the Phase Function of IIR Hilbert Transformers | 411 |
| D.1 | Calculation of the Coefficients $a(i)$ | 413 |
| D.2 | The Problem of the Value of K in (D.14) | 416 |
| Appendix E | Derivation of the Spectrum of the CSSB Signal for Linear Amplitude | 419 |
| Appendix F | Derivation of the Fourier Spectrum of the N -Dimensional Hilbert Transform | 425 |
| About the Author | | 427 |
| Index | | 429 |