



M A T H E M A T I C A L
P E R S P E C T I V E S O N
T H E O R E T I C A L P H Y S I C S

A Journey from Black Holes to Superstrings

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