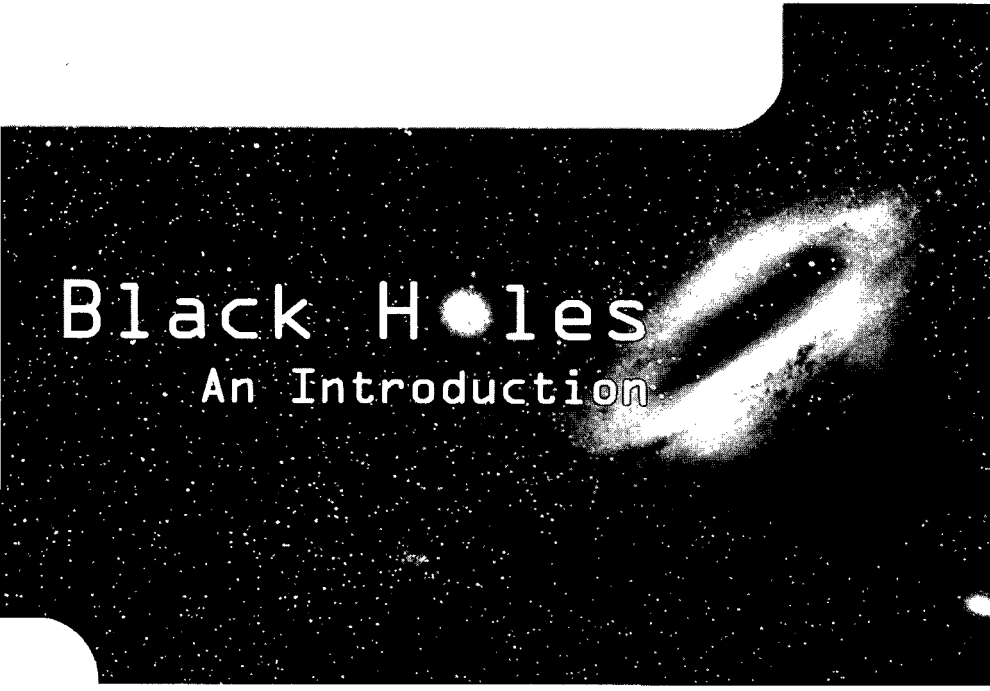

Derek Raine & Edwin Thomas
University of Leicester, UK

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