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Riemann Solvers and Numerical Methods for Fluid Dynamics

A Practical Introduction

With 223 Figures

Springer

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