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# Introduction to Mechanics and Symmetry

A Basic Exposition of  
Classical Mechanical Systems

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OF TORONTO  
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With 43 Illustrations



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