
Microstructures in Elastic Media

Principles and Computational Methods

Nhan Phan-Thien
Department of Mechanical Engineering
The University of Sydney
Australia

Sangtae Kim
Department of Chemical Engineering
University of Wisconsin-Madison
Madison, Wisconsin

New York Oxford
OXFORD UNIVERSITY PRESS
1994

Contents

1 Fundamental Equations	3
1.1 Introduction and Motivation	3
1.2 Stress and Strain	4
1.3 Equations of Equilibrium	6
1.4 Strain Energy	8
1.4.1 Uniqueness	9
1.4.2 Extremum Principles	10
1.5 Betti's Reciprocal Theorem	12
1.6 Integral Representation	13
1.6.1 Classification of Integral Equations	13
1.6.2 Kelvin State	15
1.6.3 Integral Representation	17
1.6.4 Rigid Inclusion	19
1.6.5 Eliminating Single or Double Layer	21
1.7 Single and Double Layer Potentials	22
1.7.1 Single Layer	22
1.7.2 Double Layer	25
1.7.3 Liapunov–Tauber Theorem	29
1.8 Boundary Integral Equations	29
1.8.1 Direct BEM	30
1.8.2 Indirect BEM	32
1.9 Spectral Properties	35
1.9.1 Banach's Theorem	37
1.9.2 $\lambda = -1$	37
1.9.3 $\lambda = +1$	38
1.9.4 Type II Problems	39
1.9.5 Spectral Radius of \mathcal{K}	40
1.10 Exercises	41
1.10.1 Rigid-Body Displacement	41
1.10.2 Stretching	42
1.10.3 Simple Shearing	42
1.10.4 Moduli of Elasticity	42
1.10.5 Integral Representation	42
1.10.6 Transmission of Force and Torque	43
1.10.7 Reciprocal Relation	43
1.10.8 Translating Rigid Sphere 1	43
1.10.9 Translating Rigid Sphere 2	44

1.10.10 Kelvin's Solution	44
1.10.11 On Green's Equations	45
1.10.12 Papkovich-Neuber Representation	45
1.10.13 Galerkin Vector	46
1.10.14 Self-Adjoint Property of \mathcal{G}	46
1.10.15 Elastic Inclusion	47
1.10.16 Constant c_{ij}	47
1.10.17 Thin, Rigid Inclusion	47
1.10.18 Liapunov-Tauber Theorem	47
2 Multipole Expansion and Rigid Inclusions	49
2.1 Singularity Solutions	49
2.1.1 Papkovich-Neuber Representation	49
2.1.2 Potential Deformation	51
2.1.3 Rotlet Deformation	52
2.1.4 Kelvinlet Deformation	53
2.1.5 Half-Space Solutions	55
2.1.6 Interior Deformation	58
2.2 Multipole Expansion	59
2.2.1 Stresslet	62
2.3 Spherical Rigid Inclusion	63
2.3.1 Translating a Rigid Sphere	63
2.3.2 Rotating a Rigid Sphere	65
2.3.3 Rigid Sphere in a Linear Deformation	66
2.3.4 Rigid Sphere in a Quadratic Ambient Field	70
2.3.5 Translating an Elastic Spherical Inclusion	71
2.4 Exercises	73
2.4.1 Navier Solutions	73
2.4.2 Navier Solutions	73
2.4.3 Navier Solutions	73
2.4.4 Galerkin Vector	74
2.4.5 Force and Torque on a Rigid Spherical Inclusion	74
2.4.6 Rigid Spherical Inclusion in High-Order Field	74
3 Faxén Relations and Ellipsoidal Inclusions	75
3.1 Faxén Relations	75
3.2 Rigid Spherical Inclusion	78
3.3 Rigid Ellipsoidal Inclusion	79
3.3.1 Singularity Solution for Translation	81
3.3.2 Singularity Solution for Linear Ambient Field	83
3.3.3 Degenerate Cases	86
3.3.4 Faxén Relations for the Rigid Ellipsoid	88
3.3.5 Interactions between Two Ellipsoids	88
3.4 Exercises	89
3.4.1 Traction Functionals	89
3.4.2 Faxén Relations for Torque and Stresslet	89

3.4.3 Multipole Expansion for Ellipsoids	90
3.4.4 Traction for the Translating Ellipsoid	90
4 Load Transfer Problem and Boundary Collocation	91
4.1 The Method of Reflection	92
4.2 Load Transfer between Two Spheres	93
4.2.1 Far Field by Reflection	94
4.2.2 Near Touching	99
4.3 Kelvin Solutions	102
4.3.1 Spherical Harmonics	102
4.3.2 Kelvin's General Solutions	104
4.4 Boundary Collocation	108
4.4.1 Twin Multipole Expansions	109
4.4.2 Collocation Equations for Translation Problems	110
4.5 Comparison	114
4.6 Constitutive Relation	119
4.6.1 Constitutive Theory	120
4.6.2 Cubic Lattices	122
4.7 Kelvinlet near a Rigid Sphere	124
4.7.1 The Axisymmetric Kelvinlet	126
4.7.2 The Transverse Kelvinlet	132
4.8 Exercises	137
4.8.1 Solid Spherical Harmonics	137
4.8.2 Lurié Solution	137
4.8.3 Type I Problems	138
5 Completed Double Layer Boundary Element Method	139
5.1 Introduction	139
5.2 Direct Formulation	141
5.3 Completed Double Layer Boundary Element Method	144
5.3.1 Range Completer	145
5.3.2 Null Functions of $(1 + \mathcal{K})$	146
5.3.3 Completion Process	147
5.3.4 Container Surface	149
5.3.5 A Summary	152
5.4 Rigid Inclusion	153
5.4.1 Translational Displacement	153
5.4.2 On Picard Iteration	155
5.4.3 Rotational Displacement	157
5.4.4 Homogeneous Deformation	157
5.5 Stresslet	160
5.6 Spectrum for a Sphere	161
5.6.1 Type I Problems – Ill-posed	166
5.7 Completed Double Layer Traction Problem	167
5.8 Exercises	169
5.8.1 Symmetry Relations	169

5.8.2	On Eigenfunctions	169
5.8.3	Incompressible Case	169
5.8.4	Gram-Schmidt Orthonormalization	169
5.8.5	Hadamard Ill-posed Problem	170
6	Numerical Implementation	171
6.1	Numerical Quadrature	172
6.2	Boundary Discretization	176
6.2.1	Constant Element	176
6.2.2	Higher Order Element	177
6.3	Evaluation of Boundary Integrals	182
6.3.1	Multivalued Traction	182
6.3.2	Regular Integrals	183
6.3.3	Singular Integrals	185
6.3.4	Rigid-Body Displacement	187
6.3.5	Adaptive Integration Schemes	188
6.3.6	Far-Field Approximation	191
6.4	Solution Methods	193
6.4.1	Direct Solver	193
6.4.2	Iterative Methods	193
6.4.3	Domain Decomposition	194
6.5	Distributed Computing under PVM	195
6.5.1	Some Concepts in Distributed Computing	196
6.5.2	Master/Slave Implementation	199
6.6	Exercises	201
6.6.1	Newton-Cotes rules	201
6.6.2	Quadrature	201
6.6.3	Galerkin Expansion	201
6.6.4	Jacobian	203
6.6.5	Evaluation of $\int_{\Delta} G_{ij} dS$ and $\int_{\Delta} K_{ij} dS$	203
7	Some Applications of CDL-BIEM	205
7.1	Translating Sphere	205
7.1.1	Direct Formulation	205
7.1.2	CDL-BIEM	209
7.2	Sphere in Homogeneous Deformation	212
7.3	Two Spheroids	215
7.4	CDL in Half-Space	218
7.5	Container Surface	220
7.6	Deformation of a Cluster	222
7.7	Distributed Computing under PVM	225
7.7.1	Arrays of Spheres	225
7.7.2	Epilogue: Sedimentation through an Array of Spheres	226
References		231
Index		240