

# Weakly Nonlocal Solitary Waves and Beyond-All-Orders Asymptotics

Generalized Solitons and  
Hyperasymptotic Perturbation Theory

*by*

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