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Group Theory and Its Applications in Physics

With 72 Figures

Springer-Verlag
Berlin Heidelberg New York London
Paris Tokyo Hong Kong Barcelona

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Sections marked with an asterisk may be omitted on a first reading.

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